ON THE PARALLELIZABILITY OF THE SPHERES

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(The following note consists of excerpts from two letters.)

(*Milnor to Bott*; December 23, 1957.)

. . . Hirzebruch tells me that you have a proof of his conjecture that the Pontrjagin class p_k of a GL_m-bundle over the sphere S^{4k} is always divisible by $(2k-1)!$. I wonder if you have noted the connection of this result with classical problems, such as the existence of division algebras, and the parallelizability of spheres.

According to Wu the Pontrjagin classes of any GL_m-bundle, reduced modulo 4, are determined by the Stiefel-Whitney classes of the bundle. (See *On the Pontrjagin classes* III, Acta Math. Sinica vol. 4 (1954) in Chinese.) The proof makes use of the Pontrjagin squaring operation, together with the coefficient homomorphism $i: Z_2 \to Z_4$. Although I do not know the exact formula which Wu obtains, the following special case is not hard to prove:

LEMMA. *If the Stiefel-Whitney classes $w_1, w_2, \cdots, w_{4k-1}$ of a GL_m-bundle are zero then the Pontrjagin class p_k, reduced modulo* 4, *is equal to $i_* w_{4k}$.*

For a bundle over S^{4k} this means that w_{4k} is zero if and only if p_k is divisible by 4. Now if you can prove that p_k is divisible by $(2k-1)!$ it will follow that w_{4k} must be zero, whenever $k \geqq 3$.

THEOREM. *There exists a GL_m-bundle over S^n with $w_n \neq 0$ only if n equals* 1, 2, 4 *or* 8.

PROOF. Wu has shown that such a bundle can only exist if n is a power of 2. But the above remarks show that the cases $n = 16, 32, \cdots$ cannot occur.

COROLLARY 1. *The vector space R^n possesses a bilinear product operation without zero divisors only for n equal to* 1, 2, 4 *or* 8.

PROOF. Given such a product operation the map $S^{n-1} \to GL_n$ defined by $x \to$(left multiplication by x) gives rise to a GL_n-bundle over S^n for which it can be shown that $w_n \neq 0$.

COROLLARY 2. *The sphere S^{n-1} is parallelizable only for $n-1$ equal to* 1, 3 *or* 7.

PROOF. Given linearly independent vector fields $v_1(x), \cdots, v_{n-1}(x)$,