DOMAINS OF POSITIVITY

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A Domain of Positivity D is an open convex cone associated with a nonsingular symmetric matrix S, called the characteristic, such that $x \in D$ if and only if $x'Sy > 0$ for all $y \in \bar{D}$. As such they were introduced by Koecher (1) in generalization of the cone of positive definite matrices studied by Siegel. The automorphisms of D are the nonsingular linear transformations mapping D onto itself. The group of automorphisms $\{W\}$ admits an anti-automorphism: $W \rightarrow S^{-1}W'S$, where W' means W transposed. A norm $N(x)$ is a function positive and continuous for $x \in D$ and satisfying there $N(Wx) = \|W\| N(x)$ for every automorphism W. A norm is given by:

$$1/N(x) = \int_D \exp(-x'St)dt$$

and a group invariant positive definite metric form is given by:

$$g_{ij} = -\frac{\partial^2 \log N(x)}{\partial x_i \partial x_j}\cdot$$

The Domain is called homogeneous if the automorphisms are transitive. In this case the Domain has an involution given by:

$$x \rightarrow x^* = S^{-1} \operatorname{grad} \log N(x).$$

For homogeneous domains it is easy to show that $N^2(x)$ is always a rational function. If the characteristic is positive definite much more is true. In the first instance, the fixed points of the anti-automorphism of the group of automorphisms already act transitively on the domain D. It follows that the norm satisfies the important equality:

$$N(x^* + y^*)\cdot N(x)\cdot N(y) = N(x + y).$$

Moreover, for every point x in D there is an involution of the Domain keeping x fixed. Hence D is a symmetric (Cartan) space, and it is possible to make a detailed study of the Lie group of automorphisms. The following facts emerge:

(a) $N^2(x)$ is a polynomial,

(b) The geodesic connecting any two points (given by Cartan's construction of geodesics in a symmetric space) is unique,