ON THE WHITEHEAD HOMOMORPHISM J

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Communicated by R. H. Fox, February 3, 1958

Consider the homomorphism $J: \pi_{r-1}(SO_n) \to \pi_{n+r-1}(S^n)$ of G. W. Whitehead[2] in the stable range $n > r$. The object of this note is to prove:

THEOREM 2. *Let q be an odd prime and let r be any multiple of $2(q-1)q^i$, $i \geqq 0$. Then for $n > r$ the image $J\pi_{r-1}(SO_n) \subset \pi_{n+r-1}(S^n)$ contains a cyclic subgroup of order q^{i+1}.*

According to recent work of Adams (as yet unpublished) the stable group $\pi_{n+r-1}(S^n)$ has the following q-primary components:
Z_q for $r = 2i(q-1)$, $i < q$ (this result is due to Cartan);
Z_q for $r = 2q(q-1) - 1$;
Z_{q^2} for $r = 2q(q-1)$; and zero for other values of r less than $2q(q-1)$.
Comparing this with Theorem 2 we have:

COROLLARY. *For $r < 2q(q-1) - 1$, and for $r = 2q(q-1)$, the image $J\pi_{r-1}(SO_n)$ contains the q-primary component of the stable group $\pi_{n+r-1}(S^n)$.*

The corresponding assertion for $r = 2q(q-1) - 1$ is false, since the group $\pi_{r-1}(SO_n)$ is zero[3] in this case.

The proof will be based on work of Thom, Hirzebruch, Borel and von Staudt.

THEOREM 1. *Let ξ be the SO_n-bundle over S^r corresponding[4] to an element λ of $\pi_{r-1}(SO_n)$. If $J\lambda = 0$ then there exists an oriented manifold M^r, differentiably imbedded in the sphere S^{n+r}, and having the following property: Some map $g: M^r \to S^r$ of degree $+1$ is covered by a bundle map of the normal bundle of M^r into the given bundle ξ.*

(It is not asserted that M^r is connected.) The manifolds M^r constructed in this way will be further studied in a later paper.[5]

PROOF OF THEOREM 1. Let E be the total space of the n-cell bundle over S^r associated with ξ; so that the boundary $\dot{E}$ is the total space

[1] The author holds a Sloan fellowship.

[2] G. W. Whitehead, Ann. of Math. vol. 43 (1942), pp. 634–640.

[3] See R. Bott, Proc. Nat. Acad. Sci. U.S.A. vol. 43 (1957) pp. 933–935.

[4] See Steenrod, *The topology of fibre bundles*, 1951, p. 99.

[5] J. Milnor, *A generalization of a theorem of Rohlin*, to appear.