RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

PRODUCTS OF SYMMETRIES

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A (bounded) operator Q on a (complex) Hilbert space H is a *symmetry* if it is a unitary involution, i.e., if $Q^*Q=QQ^*=1$ (=the identity operator on H) and $Q^2=1$. In connection with his studies of the infinite-dimensional analogues of the classical groups, R. V. Kadison has asked us which operators can be represented as (finite) products of symmetries. The purpose of this note is to give a precise answer to Kadison's question.

THEOREM 1. *If H is infinite-dimensional, then every unitary operator on H is the product of four symmetries.*

PROOF. We need the auxiliary result that if U is a unitary operator on an infinite-dimensional Hilbert space H, then there exists a (closed) subspace H_0 of H such that H_0 reduces U and such that $\dim H_0 = \dim H_0^\perp$. This result holds, in fact, for an arbitrary normal operator on H. Since the proof is a straightforward application of the spectral theorem, and since the proof for a typical special case (namely, for Hermitian operators) has already appeared in the literature,[1] we do not present it here.

We apply the auxiliary result to the (unitary) operator on $H_0^\perp$ obtained by restricting U to $H_0^\perp$ and obtain thus a subspace H_1 (of $H_0^\perp$) such that H_1 reduces U and such that $\dim H_1 = \dim (H_0^\perp \cap H_1^\perp)$. Proceeding inductively, we obtain an infinite sequence $\{H_n\}$ of orthogonal subspaces (of H) such that each H_n reduces U and such that every H_n has the same dimension. If the intersection of the orthogonal complements of all the H_n is not trivial, it can be amalgamated to H_0; it follows that H is the direct sum of countably many equidimensional subspaces each of which reduces H. By suitably renumbering the terms of this sequence, we may assume that the index n runs through all (not necessarily non-negative) integers.

Relative to the fixed direct sum decomposition $H = \sum_n H_n$, we

[1] Paul R. Halmos, *Commutators of operators*, Amer. J. Math. vol. 74 (1952) pp. 237–240; see Lemma 3 on p. 239.