BOOK REVIEWS

The geometry of geodesics. By Herbert Busemann. New York, Academic Press, 1955. 10+422 pp. $9.00.

This book represents a most interesting and successful effort to treat in a unified fashion the geometry of a class of metric spaces general enough to include Riemannian and Finsler spaces. Only the barest minimum is postulated; in particular, the spaces are metric; bounded infinite sets have limit points; any two points can be connected by a (geodesic) segment, i.e. a curve whose length equals the distance of the points; every point has a neighborhood in which a geodesic segment can be prolonged in a unique way. It is a consequence that a given segment can be prolonged indefinitely in both directions to yield a geodesic, i.e. a curve which is locally a segment. These are more or less paraphrases, the five axioms themselves being stated in terms of the metric, and in very simple form. The spaces satisfying them are called G-spaces and are the object of study throughout the book. They include the complete Riemannian and Finsler manifolds. It should be pointed out however that it is not assumed that the spaces are differentiable, or even topological manifolds. In fact, it is an important unsolved question of the theory as to whether the axioms imply this. It is shown that the dimension, which may, so far as is known, be infinite, is the same at each point, and that should it be two, then the space is a manifold. It is a remarkable fact that in many instances it is shown that the addition of a single property or axiom is enough to characterize well-known geometries, of various types. For example if $B(x, y)$ denotes the bisector (set of points equidistant from) x and y, then the assumption that bisectors are flat, i.e. contain with each pair of points a segment joining them, implies that the space is euclidean, hyperbolic or spherical and of dimension greater than one. This implies a second characterization of these same spaces as solutions to the Helmholtz-Lie problem, to wit: if for any two isometric point triples of a G-space a motion exists which carries the first into the second, the same conclusion as above follows.

In all there are six chapters; the first deals with general concepts and contains as a highlight a solution in the large of the inverse problem of the calculus of variations for a system of curves in the plane: if each curve of the system goes to infinity at each end and any two points lie on exactly one curve, then the plane may be metrized as a G-space with these curves as geodesics. Chapter II is devoted to G-