BOOK REVIEWS

Analytische Fortsetzung. By L. Bieberbach. (Ergebnisse der Mathematik und ihrer Grenzgebiete, N.S., no. 3.) Berlin, Springer, 1955. 4+168 pp. 24.80 D.M.

The general scope of this monograph is approximately the same as the Chapter of the same title in the author's *Lehrbuch der Funktionentheorie* (Chap. 7, vol. II, Teubner, 1931) brought up to date by the developments of the last 25 years. Much of this was initiated by Pólya and those who have been influenced by him. The central problem can be stated somewhat as follows: how are properties of the coefficient sequence $\{a_n\}$ reflected in the behavior of $\sum a_n z^n$, and conversely? Three chief tools form the basis for the development, and are discussed in Chapter 1; these are the method of associated functions, linked with the Laplace-Borel transform, the Hadamard multiplication theorem, and the Euler series transformation. If $A(z) = \sum_0^\infty c_n z^n/n!$ is an entire function of exponential type, its Borel transform is $a(z) = \sum_0^\infty c_n/z^{n+1}$. The rate of growth of A along rays is closely connected with the region of regularity of $a(z)$; in turn, one may immediately obtain information about the location of singularities of the associated function $g(z) = \sum_0^\infty A(n)z^n$. The classical multiplication theorem of Hadamard deals with the three power series, $a(z) = \sum a_n z^n$, $b(z) = \sum b_n z^n$, and $c(z) = \sum a_n b_n z^n$. An open set S is a star (at 0) if $\lambda S < S$ for all λ, $0 \leqq \lambda \leqq 1$. If A and B are star sets, their star-product is $A \odot B = (A' \cdot B')'$. If $a(z)$ is regular for $z \in A$, and $b(z)$ for $z \in B$, then $c(z)$ is regular at least in $A \odot B$. It is not true [as other books have not always pointed out] that the singularities of $c(z)$ necessarily have the form $\alpha\beta$ where α and β are, respectively, singularities of $a(z)$ and $b(z)$. However, every suitably restricted boundary point of $A \odot B$ which is a singularity for $c(z)$ has this form. The Euler transformation provides a simple necessary and sufficient condition that a power series $\sum a_n z^n$ have 1 as a singularity.

It is difficult to give more than an indication of the skill and elegance with which the author combines these simple and familiar tools to obtain a unified treatment of a selected portion of the theory of Taylor series. In Chapters 2 and 3, one finds a discussion of the effect of coefficient gaps upon the presence and location of singularities of a power series, starting with the general theorems of Fabry and Pólya, and ending with more detailed and specialized theorems of Wilson, MacIntyre, Mandelbrojt and many others. In particular, §2.1 is devoted to a clarification [with enlightening comments] of Fabry's