BOOK REVIEWS

Eine neue Methode in der Analysis und deren Anwendungen. By Paul Turán. Budapest, Akadémiai Kiadó, 1953. 195+1 pp. and errata. 80.00 Ft.

This is a connected account, with extensions, of a method developed by the author in a series of papers beginning in 1941. The general theme is the application of Diophantine approximation to analysis and analytical theory of numbers. In the extensive literature that has grown up round this subject during the present century, arithmetic and analysis have become linked in a mutual relationship. Arithmetical theorems on the simultaneous approximation to real numbers by integers have been applied to the proof of inequalities for trigonometrical sums

$$f(t) = \sum_{j=1}^{k} a_j e(\lambda_j t) \qquad [\lambda_j \text{ real}; e(x) = e^{2\pi i x}];$$

and, conversely, arithmetical theorems have been derived from inequalities for suitably chosen sums of this type. Thus, the classical theorems of Dirichlet and Kronecker stand in this mutual relationship to theorems about the solubility in t of the inequality $|f(t)| > \theta f^*(0)$ for a given $\theta < 1$, where $f^*(t)$ is $f(t)$ with a_j replaced by $|a_j|$. For Dirichlet's theorem the appropriate $f(t)$ have a_j real and positive, but to compensate for this restriction the solutions t can be "localized"; thus, if $\theta = \cos(2\pi/\omega)$, where $\omega > 4$, there is a solution in any given interval $\tau \leqq t \leqq \tau\omega^k$ $(\tau > 0)$. For Kronecker's theorem no restriction is placed on the a_j, but the λ_j must be supposed linearly independent (over the field of rational numbers); and there is no localization, except in special cases where the degree of linear independence can be estimated quantitatively (as, for example, when the λ_j are logarithms of primes).

By way of introducing his own point of view the author makes two comments on this situation; firstly, that the analytical theorems have equal right with the arithmetical to be considered fundamental; and, secondly, that for many purposes close localization is more useful than a strong inequality. He is thus led to formulate, as basic tools for direct application, a series of inequalities for a generalized $f(t)$ with complex λ_j. The enunciations involve only integral values of t, and for such values we may write unambiguously, after rearrangement of the terms,