SEMI-GROUPS OF OPERATORS

R. S. PHILLIPS[1]

1. **Introduction.** I should like to present some of the recent developments in the theory of one-parameter semi-groups of linear bounded operators. Actually this field is relatively new, having its origin in the work of M. H. Stone [16] on one-parameter groups of unitary operators in Hilbert space which appeared in 1930. By now the subject has an extensive literature, the definitive work being E. Hille's [4] *Functional analysis and semi-groups*, published in 1948. An adequate bibliography for the material in this field published prior to 1948 can be found in Hille's treatise. I shall limit my remarks to developments since 1948, emphasizing perhaps unduly my own contributions. Further I shall not discuss in any detail the applications to partial differential equations of parabolic type, although this has been one of the most gratifying aspects of the theory; this material can be found in papers by W. Feller [1], E. Hille [7], and K. Yosida [20; 21].

Let $\mathfrak{S} \equiv [T(\xi); \xi > 0]$ be a one-parameter family of linear bounded operators defined on a complex Banach space $\mathfrak{X}$ to itself and satisfying the product law

$$T(\xi_1 + \xi_2) = T(\xi_1)T(\xi_2), \qquad \xi_1, \xi_2 > 0. \tag{1.1}$$

We shall further assume that $T(\xi)x$ is a continuous function of ξ for $\xi > 0$ and each $x \in \mathfrak{X}$, that is, we assume that $\mathfrak{S}$ is continuous in the strong operator topology. The *infinitesimal operator* is defined as the limit in norm

$$\lim_{\eta \to 0+} \frac{T(\eta) - I}{\eta} x = A_0 x \tag{1.2}$$

wherever this limit exists, the domain of A_0 (in symbols $\mathfrak{D}(A_0)$) being the set of elements for which the limit exists. It is easy to see that A_0 is a linear operator. However, A_0 need not be bounded nor even closed; its least closed extension, when it exists, will be called the *infinitesimal generator of* $\mathfrak{S}$ and will be designated by A. The latter operator plays a basic role in the theory. For $x \in \mathfrak{D}(A_0)$ we have

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