MEASURE AND AREA

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1. **Introduction.** For a long time the efforts of many mathematicians have been directed toward the creation of a complete theory which extends the quantitative concepts of Euclidean geometry to general subsets of Euclidean n-space, E_n, and to mappings into E_n. The first and fundamental steps in the modern development of this subject were taken by Lebesgue and Carathéodory with the invention of the general theory of measure. Subsequently the main interest has shifted naturally to the specific geometric problems which arise when the new concepts are applied in the extension of classical analytic formalisms and in the structural study of sets and mappings.

Measures are functions defined on classes of point sets; areas are functions defined on classes of mappings.

The first four sections of this paper are mainly expository. They contain the definitions of certain measures of geometric interest; a discussion of the rectifiability, tangent planes, densities and projections of point sets; and a description of a very general form of the Gauss-Green Theorem which illustrates the scope of the theory and its applicability to classical problems. Pivotal in this whole structure is the relationship between the purely metric notion of Hausdorff measure and the group-theoretic concepts of integral geometry.

The remaining four sections of this paper serve to establish some new results concerning the area of mappings. The basic issue motivating these investigations is the validity of the principle that every area defined by any reasonable method is naturally representable as an integral of a multiplicity function with respect to a measure. This raises a difficult problem because Lebesgue, in introducing his particular area which has been used very successfully by many workers in the field, was guided entirely by the analytic requirement of semi-continuity; his definition appears to have no immediate connection with the geometric measure theory which was created later by Carathéodory. This difficulty has now been overcome for the case of prime classical interest: mappings of a two-cell into three-space.

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