ERGODIC SETS

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Introduction. Ergodic sets were introduced by Kryloff and Bogoliouboff in 1937 in connection with their study of compact dynamical systems [16]. The purpose of this paper is to review some of the work that has since been done on the theory that centers around this notion, and to present a number of supplementary remarks, applications, and simplifications. For simplicity we shall confine attention to systems with a discrete time. Continuous flows present no difficulty, but the development of a corresponding theory for general transformation groups is still in an incomplete stage. An example due to Kolmogoroff (see [5]) shows that such an extension cannot be made without sacrificing either the invariance or the disjointness of ergodic sets.

In §§1 and 2 we give a brief, but self-sufficient, development of the basic theorems of Kryloff and Bogoliouboff. In §3 we collect some auxiliary results for later use. In §4 a simple characterization of transitive points is obtained. In §5 the distinctive properties of some special types of systems and subsystems are discussed, and in §6 these results are used to discover conditions under which the ergodic theorem holds uniformly. In §7 a generalization to noncompact systems is considered, and in §§8 and 9 some known representation theorems are obtained as an application of ergodic sets. In §10 there is given an example of a minimal set that is not strictly ergodic, similar to one constructed by Markoff.

1. Some corollaries of the ergodic theorem. We shall use the following notations: If $f(p)$ is a real-valued function on a set Ω, and if T is a 1:1 transformation of Ω onto itself, then

$$M(f, p, k) = f_k(p) = \frac{1}{k}\sum_{i=1}^{i=k} f(T^i p) \quad (k = 1, 2, \cdots)$$

and

$$M(f, p) = f^*(p) = \lim_{k\to\infty} M(f, p, k)$$

in case this limit exists. The characteristic function of a set E is

An address delivered before the Washington meeting of the Society, October 27, 1951, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors November 12, 1951.