THE ASYMPTOTIC DENSITY OF SEQUENCES

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1. **Introduction.** Our purpose is to outline the recent work on the asymptotic or limit density of sets of positive integers, and to give further details of some recently announced results [45; 46].[1] §§2–7 are concerned with the first objective, and §§8–12 with the second. The related concept of Schnirelmann density is touched upon, but we mention only the high spots of work on this topic, including basic sequences and essential components.

In the case of many sequences, much more than the mere density is known. The prime number theorem, for example, implies that the set of primes has density zero, but it tells much more. However, asymptotic estimates are not available for many sequences, and for these it is of interest to know the density. In selecting examples of sequences of density zero in §11, as applications of the results of §8, we have tried to choose sets for which analytic estimates are not at present known. We are indebted to H. S. Zuckerman for valuable suggestions concerning the formulations in §8.

2. **Definitions.** The sequence A of positive integers $a_1 < a_2 < \cdots$ has lower density $\delta_1(A)$ and upper density $\delta_2(A)$ defined by

$$\delta_1(A) = \liminf_{n\to\infty} \frac{A(n)}{n}, \qquad \delta_2(A) = \limsup_{n\to\infty} \frac{A(n)}{n}, \tag{1}$$

where $A(n)$ denotes the number of integers of A which are not greater than n. The value $\delta_1(A)$ has been referred to variously as the asymptotic density, limit density, or density of A. In this paper, however, we shall say that A has a density $\delta(A)$ only if $\delta_1(A) = \delta_2(A)$, in which case we can write

$$\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}. \tag{2}$$

This is sometimes called the natural density of A. The assertion that almost all positive integers have a certain property P means that $\delta(A) = 1$, where A is the set of integers having the property P. It can be established that for infinite sequences $\delta_1(A) = \liminf n/a_n$,

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[1] Numbers in brackets refer to the references at the end of the paper.