RIEMANN'S METHOD AND THE PROBLEM OF CAUCHY

MONROE H. MARTIN

Introduction. The method devised by Riemann to solve the Problem of Cauchy applies to linear, hyperbolic, partial differential equations of second order for one unknown function u of two independent variables x, y. For a homogeneous equation the essential points in the method are:

(a) The introduction of the characteristics as coördinate lines,

(b) The construction of a line integral $I=\int\{Bdx-Ady\}$ which vanishes around closed paths, where:

(c) A and B are certain bilinear forms in u, u_x, u_y and v, v_x, v_y, and,

(d) v (Riemann's function) is a properly chosen two-parameter family of solutions of a second linear partial differential equation, the *adjoint equation.*

In this paper new bilinear forms are taken for A, B in (b), and the rôle of the adjoint equation in (d) is taken over by a partial differential equation termed the *associate equation* of the original equation. Each solution ϕ of the original equation leads to an associate equation and the Problem of Cauchy is then solved with the aid of a properly chosen two-parameter family of solutions of the associate equation called the *resolvent,* the analogue of Riemann's function.

This modification offers some hope of extending Riemann's method to the Problem of Cauchy for linear hyperbolic partial differential equations with more than two independent variables.[1] Such an extension to three independent variables is actually carried out in this paper for the equation of cylindrical waves.[2] In the treatment of this equation, following (a), characteristic coördinates are adopted. As an interesting corollary, it turns out that axially symmetric solutions are governed by one[3] of Euler's partial differential equations, namely,

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[1] The classic work on Cauchy's problem is the book by J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperbolique*, Paris, 1932, where references to the work of other authors are given.

[2] My colleague A. Weinstein has kindly pointed out that an extension of Riemann's method for this equation has been given by H. Lewy, *Verallgemeinerung der Riemannschen Methode auf mehr Dimensionen*, Nachr. Ges. Wiss. Göttingen (1928) pp. 118–123. His method differs from ours in that it employs three "Riemann functions," and neither makes use of characteristic coördinates, nor of an "associate equation."

[3] See, for example, G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, 2d ed., Paris, 1915, pp. 54–70.