

## ON THE DIFFERENTIAL GEOMETRY OF CLOSED SPACE CURVES

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1. The subject of this lecture, the study of the differential geometric properties of a space curve which depend on the assumption that the curve is closed, is a rather modest chapter of the differential geometry in the large. The results are often comparatively elementary and seem to be isolated. On the other hand, the intuitive character of the statements and the lack of a general method of approach make the field attractive, and in the latest years several authors have contributed to it. Therefore, it might be justified to give a survey of its actual state. In an attempt to gather some results and problems from a common point of view it turned out that the differential geometry of space curves becomes more satisfactory if it is developed under slightly weaker assumptions than those usually adopted. Furthermore it proves to be useful to attach more importance to the simple geometrical relations between the spherical indicatrices of a curve and to the kinematical interpretation initiated by G. Darboux. Though many of the results are or can be generalized to curves in a euclidean space of arbitrary dimension, the following exposition is restricted to the case of the ordinary space, and it will not be possible to mention all contributions to the subject.

2. Let  $s$ ,  $0 \leq s \leq l$ , denote the arc length and  $\mathbf{r}(s)$  the position vector of a variable point of a space curve  $K$ . Differentiation with respect to  $s$  is indicated by a prime. Our assumptions are the following: The coordinates of  $\mathbf{r}(s)$  are functions of class 4. To every point  $\mathbf{r}(s)$  is attached an osculating plane, that is, a plane containing the vectors  $\mathbf{t} = \mathbf{r}'$  and  $\mathbf{r}''$  such that its suitably oriented normal unit vector  $\mathbf{b}(s)$ , the binormal vector of  $K$ , has coordinates which are functions of class 2.<sup>1</sup> The vectors  $\mathbf{t}'$  and  $\mathbf{b}'$  do not vanish simultaneously. For the sake of simplicity we further assume that they vanish only at a finite number of points. This implies that no arc of  $K$  is contained in a plane.

By this formulation we avoid the usual assumption that  $\mathbf{r}''$  and,

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<sup>1</sup> Actually we are given an osculating strip in the sense of Blaschke [1, p. 72]. (Numbers in brackets refer to the references at the end of the paper.)