DUALITY FOR GROUPS

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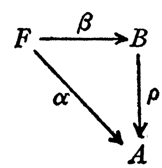
I. THE PHENOMENON OF DUALITY

1. Abelian groups. Certain dualities arise in those theorems of group theory which deal, not with the elements of groups, but with subgroups and homomorphisms. For example, a free abelian group F may be characterized in terms of the following diagram of homomorphisms:

(1.1)



THEOREM 1.1. *The abelian group F is free if and only if, whenever ρ: $B \to A$ is a homomorphism of an abelian group B onto an abelian group A and α: $F \to A$ a homomorphism of F into A, there exists a homomorphism β: $F \to B$ with*

$$\rho\beta = \alpha. \tag{1.2}$$

If F is known to be free, with generators g_i, β may be constructed by setting $\beta g_i = b_i$, with b_i so chosen that $\rho b_i = \alpha g_i$. Conversely, let F have the cited property and represent F as a quotient group F_0/R_0, where F_0 is a free abelian group. Choose $A = F$ and $B = F_0$ in (1.1), let α be the identity, and ρ the given homomorphism of F_0 onto F with kernel R_0. Then, by (1.2), $\alpha = \rho\beta$ is an isomorphism, hence β has kernel 0 and thus is an isomorphism of F into F_0. Therefore F is isomorphic to a subgroup of a free group F_0, so is itself free.

The analogous theorem is true for free nonabelian groups, when A and B are interpreted as arbitrary (not necessarily abelian) groups; the proof uses the Schreier theorem [**14**][2] that a subgroup of a free group is free.

An abelian group D is said to be *infinitely divisible* if for each $d \in D$ and each integer m there exists in D an element x such that $mx = d$. Such groups may be characterized by a similar diagram

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[2] Numbers in brackets refer to the bibliography at the end of the paper.