BOOK REVIEW

Divergent series. By G. H. Hardy. Oxford University Press, 1949. 16+396 pp. $8.00.

Hardy died on December 1, 1947; during his lifetime the theory of divergent series and its applications developed into an important branch of modern analysis. It is only natural that the book bears all the marks of his own research work, but it is also a comprehensive presentation of this vastly expanded subject. The final galley proofs were read and completed by some of his younger collaborators, in particular L. S. Bosanquet.

The book consists of thirteen chapters and five additional sections. Attached to each chapter are notes and references. Chapters I and II contain a lively historical survey, particularly on Euler's, Fourier's, and Heaviside's contributions. Also some principles are discussed on which to base methods for the summation of divergent series and integrals.

Chapter III discusses general theorems concerning linear transformations of sequences and functions, with some particular examples. Let us explain a few fundamental concepts. The general linear transformation of a sequence s_n is either another sequence

$$t_m = \sum_{n=0}^{\infty} c_{m,n}s_n, \qquad m = 0, 1, 2, \cdots,$$

or a function $t(x) = \sum c_n(x)s_n$, where x is a continuous parameter. A linear transform of a function $s(y)$ is $t(x) = \int_0^\infty c(x, y)s(y)dy$; $(c_{m,n})$ is the matrix of the transform t_m, $c(x, y)$ is the kernel of the transform $t(x)$.

In Chapter IV special methods are discussed (Nörlund, Euler, Abel, and others). Chapter V is concerned with Hölder, Cesàro, and Riesz means. Hölder and Cesàro means of first order, denoted by $(H, 1)$ and $(C, 1)$, are the same: $h'_n = (1/(n+1)) \sum_0^n s_\nu$; by iteration: $h_n^{(2)} = (1/(n+1)) \sum_0^n h'_\nu$, and $h_n^{(r)} = (1/(n+1)) \sum_0^n h_\nu^{(r-1)}$. The Cesàro means are defined by taking $s'_n = \sum_0^n s_\nu$, $s_n^{(r)} = \sum_0^n s_\nu^{(r-1)}$, $c_n^{(r)} = s_n^{(r)}/A_n^{(r)}$, where $A_n^{(r)}$ is $s_n^{(r)}$ when all $s_n = 1$.

The next two chapters are devoted to Tauberian theorems for Cesàro and for Abel means. Tauber proved in 1897 that Abel summability and $\lim na_n = 0$ imply convergence of $\sum a_n$; this elementary result was the starting point of a long chain of investigations to establish inverse theorems of summability dealing with the question: