SOME CLASSES OF FUNCTIONS DEFINED BY DIFFERENCE OR DIFFERENTIAL INEQUALITIES[1]

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1. Introduction. In the year 1934[2] I introduced classes of functions which I called monotonic of order n. The class corresponding to $n=1$ represents the functions which are monotonic in the ordinary sense. The monotonic functions of higher order are closely connected with some distinguished functions of matrices whose independent and dependent variables are real symmetric matrices of the same order.

In a recent investigation I found that the study of monotonic functions can be subordinated to the consideration of more general classes of functions which have distinguished *group theoretical* properties. I wish to give an account of this study.

All functions considered are supposed to be *real functions of a real variable*. To avoid unessential complications, I shall consider them defined *in open intervals and continuous and strictly monotonically increasing there*. A class S of such functions will be called a *transformation semigroup* if it has the following properties:

(a) If $f(x) \in S$ and is defined in an interval (a, b), then $f(x)$, considered only in a subinterval $(a', b') \subset (a, b)$, should also belong to S.

(b) If $f(x) \in S$ and $g(x) \in S$ and the composition $g[f(x)]$ can be performed, that is, if the range of $f(x)$ falls into the domain of $g(x)$, then $g[f(x)] \in S$.[3]

(c) If $f(x)$ is a limit of functions $f_n(x) \in S$ $(n=1, 2, 3, \cdots)$, all functions of the sequence being defined in the same interval (a, b) and the convergence being uniform in any closed bounded subinterval of (a, b), then $f(x) \in S$. Condition (c) represents a closure condition on S.

In the following the expression *transformation group* shall mean a class of functions which in addition to (a), (b), (c) satisfy condition

(b′) If $f(x) \in S$, then the inverse $f^{-1}(x) \in S$.

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[1] The results of §§3 and 4 were obtained while the author was under contract with the Office of Naval Research N60NR248.

[2] Löwner, Karl, *Über monotone Matrixfunctionen*, Math. Zeit. vol. 38 (1934).

[3] Because we do not assume that any two functions of S may be composed, we might speak also of a semigroupoid. We prefer the simpler expression semigroup.