RAMIFICATIONS, OLD AND NEW, OF THE EIGENVALUE PROBLEM

HERMANN WEYL

Since this is a lecture dedicated to the memory of Josiah Willard Gibbs let me start with that purely mathematical discovery which Gibbs contributed to the theory of Fourier series. Fourier series have to do with the eigenvalues and eigenfunctions of the oldest, simplest, and most important of all spectrum problems, that of the vibrating string. In preparing this lecture, the speaker has assumed that he is expected to talk on a subject in which he had some first-hand experience through his own work. And glancing back over the years he found that the one topic to which he has returned again and again is the problem of eigenvalues and eigenfunctions in its various ramifications. It so happens that right at the beginning of my mathematical career I wrote two papers on what we now call the Gibbs phenomenon.

1. **Gibbs phenomenon.** Take a simple periodic function with a discontinuity, for example, the function $1^0(x)$ of period 2π which equals 0 for $-\pi<x<0$ and 1 for $0<x<\pi$. In a letter to the editor of Nature published on April 27, 1899, Gibbs, correcting a statement in a previous letter, pointed out that the limit of the graphs of the partial sums $y=1_n^0(x)$ of the Fourier series of $1^0(x)$ includes not only the vertical ascent from the level 0 to the level 1 at $x=0$, but extends vertically beyond it by a specific amount. A. A. Michelson had started the discussion in Nature by criticizing the way in which the mathematicians are wont to describe the limit of the sequence of those partial sums; he had pleaded for adding to the two horizontal levels the vertical precipice. Today we find in the notion of uniform convergence the most adequate analysis of the phenomenon. Introduce the sinus integral

$$\mathrm{Si}\,(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{\sin \xi}{\xi}\, d\xi$$

and consider a closed interval I, say $-\pi/2\leqq x\leqq\pi/2$, containing only the one discontinuity at $x=0$. It is, of course, not true that the difference between $1^0(x)$ and the nth partial sum $1_n^0(x)$ converges

The twenty-second Josiah Willard Gibbs lecture delivered at Columbus, Ohio, December 28, 1948, under the auspices of the American Mathematical Society; received by the editors January 17, 1949.