ON INTERPOLATION TO AN ANALYTIC FUNCTION IN EQUIDISTANT POINTS: PROBLEM β

J. L. WALSH AND W. E. SEWELL

The present note is an addendum to a paper by the same authors,[1] to which (especially §8) the reader may refer for detailed notation and definitions.

Two phases of the direct problem of the study of degree of convergence of a sequence of functions approximating to a given function are (i) proof of the existence of a sequence approximating with a certain degree of convergence and (ii) study of the degree of convergence of a sequence defined by a specific method. We are here concerned with the second phase of the problem:

THEOREM 1. *Let the function $f(z)$ be analytic and of class $L(p, \alpha)$ in the annulus $\gamma_\rho: \rho > |z| > 1/\rho < 1$, where ρ is given. Let*

$$p_n(z) = a_{nn}z^n + a_{n,n-1}z^{n-1} + \cdots + a_{n0} + \cdots + a_{n,-n}z^{-n}$$

be the unique polynomial in z and $1/z$ of degree n which coincides with $f(z)$ in the $(2n+1)$st roots of unity. Then for z on $\gamma: |z| = 1$ we have

$$(1) \qquad |f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha},$$

where M is independent of n and z.

The polynomial $p_n(z)$ may be defined by the equation

$$(2) \qquad f(z) - p_n(z) = \frac{1}{2\pi i}\int_{|z|=r} + \int_{|z|=1/r} \frac{t^n(z^{2n+1} - 1)f(t)dt}{z^n(t^{2n+1} - 1)(t - z)},$$

for z in the annulus $1 > r > |z| > 1/r$; the function $f(z)$ is assumed analytic for $r > |z| > 1/r$, continuous for $r \geqq |z| \geqq 1/r$. In particular, if $f(z)$ itself is here chosen as a polynomial $P_n(z)$ in z and $1/z$ of degree n, the interpolating polynomial $p_n(z)$ must coincide with $P_n(z)$, so we have for $r > |z| > 1/r$

$$(3) \qquad 0 = \frac{1}{2\pi i}\int_{|t|=r} + \int_{|t|=1/r} \frac{t^n(z^{2n+1} - 1)P_n(t)dt}{z^n(t^{2n+1} - 1)(t - z)}.$$

Return to the original function $f(z)$ in (2) with subtraction of (3) from (2) thus yields for $r > |z| > 1/r$

Received by the editors September 24, 1948.

[1] Trans. Amer. Math. Soc. vol. 49 (1941) pp. 229–257.