HYPERGEODESIC CURVATURE AND TORSION

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1. Introduction. A *hypergeodesic curve*[1] on a surface satisfies an ordinary second-order differential equation similar to that for the geodesic curves except that the coefficients, instead of being Christoffel symbols, are taken to be arbitrary functions of the surface coordinates. At a point of the surface the envelope of all the osculating planes of hypergeodesic curves of a family determined by such an equation which pass through the point will be a cone usually of order four and class three with its vertex at the point. If the cone at each point degenerates into a single line of a congruence of lines, then the family of hypergeodesics is the family of *union curves*[2] relative to this congruence. Recently, a union curvature[3] and a union torsion[4] have been defined relative to a family of union curves as generalizations of geodesic curvature and geodesic torsion.

The following investigation deals with the differential equation for an arbitrary family of hypergeodesic curves on a surface and the definition of a hypergeodesic curvature and a hypergeodesic torsion relative to a family of hypergeodesic curves in such a manner that the definitions will reduce to those for union curvature and union torsion when the family is taken to be a family of union curves. A geometric interpretation of hypergeodesic curvature is given which generalizes the geometric interpretation for union curvature and geodesic curvature. Finally, a geometric condition that a hypergeodesic be a plane curve is obtained as a generalization of a theorem known for union curves.

The notation of Eisenhart[5] is used throughout except that $\Gamma^{\alpha}_{\beta\gamma}$ is used for Christoffel symbols of the second kind with respect to the coefficients of the first fundamental form for the surface. The summation convention of tensor analysis with regard to repeated indices

Presented to the Society, April 30, 1949; received by the editors September 16, 1948.

[1] E. P. Lane, *Projective differential geometry*, University of Chicago Press, 1942, p. 192.

[2] P. Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, Amer. J. Math. vol. 40 (1918) p. 213.

[3] C. E. Springer, *Union curves and union curvature*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 686–691.

[4] C. E. Springer, *Union torsion of a curve on a surface*, Amer. Math. Monthly vol. 54 (1947) pp. 259–262.

[5] L. P. Eisenhart, *Differential geometry*, Princeton University Press, 1940.