ON MINKOWSKI BODIES OF CONSTANT WIDTH

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A metric set is entire if the addition of any point to the set increases the diameter. A convex body has constant width if all pairs of parallel supporting planes are the same distance apart. These concepts are known to be equivalent in euclidean space.[1] The present paper shows that they are also equivalent in a minkowski space.

A proof for this equivalence for the minkowski plane was given by Meissner.[2] He showed also that two curves of the same constant width have the same circumference, and that a three-dimensional body of constant width has plane sections of constant width, in terms of the corresponding section of the minkowski sphere as plane indicatrix. However, Meissner's three-space proof for the equivalence of entireness to constant width is incomplete.[3] He assumed, moreover, that the indicatrix had no singular points. The equivalence here is shown for the n-dimensional case with the assumption merely that the indicatrix is convex.

In a euclidean E^n space let C be a closed, convex hypersurface with O as center. In terms of C as indicatrix let a minkowski distance be defined in the usual way to make the space an M^n.[4]

Let r and s be half-rays emanating from O at an angle θ and let λ_r, λ_s be the euclidean lengths of the minkowski radii of C in these directions. Then $\mathrm{sm}(r, s)$, a positional sine with respect to C, is defined to mean $\lambda_r\lambda_s \sin \theta$.[5]

LEMMA. *If r, s, t are half-rays through O, which lie in a plane, with s between r and t (in terms of an angle not greater than π) then* $\mathrm{sm}(r, s) + \mathrm{sm}(s, t) \geq \mathrm{sm}(r, t)$.

If X_1, X_2, X_3 are the end points of radii λ_r, λ_s, λ_t, then from the convexity of C it follows that the euclidean area of ΔOX_1X_2 plus the area of ΔOx_2x_3 is not less than the area of ΔOx_1x_3. Since the area of

Received by the editors August 16, 1948.

[1] Bonnesen and Fenschel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik, 1934, p. 128. Jessen, *Über konvexe Punktmengen konstanter Breite*, Math. Zeit. vol. 29 (1928) pp. 378–380.

[2] E. Meissner, *Über Punktmengen konstanter Breite*, Vierteljahrschrift der Naturforschenden Gesellschaft, 1911.

[3] This was already noticed in the references under footnote 1.

[4] Bonnesen and Fenschel, cf. footnote 1, p. 23.

[5] This was taken from a more general minkowski sine function defined by H. Busemann who has in preparation a paper on the subject.