A NOTE ON THE UNIQUE FACTORIZATION OF ABSTRACT ALGEBRAS

FREDERICK B. THOMPSON

In their monograph[1] *Direct decompositions of finite algebraic systems*, Jónsson and Tarski have established the unique factorization theorem and several related results for a comprehensive class of finite abstract algebras. Every algebra $\mathfrak{A}$ of this class is constituted by a set A of arbitrary elements, by a binary operation $+$, and possibly by other operations 0_0, 0_1, $\cdots$; the only conditions imposed on $\mathfrak{A}$ are that A be closed under all the operations involved and that it contain an element z which is a (both-hand) zero element for the operation $+$ and is idempotent under the remaining operations. At the end of their paper, Jónsson and Tarski raised the problem of whether their results can be extended to an even more comprehensive class of algebras, in fact to algebras $\mathfrak{A}$ differing from those mentioned above in that the element z is only to be idempotent under all operations involved (not necessarily being a zero element for $+$). The purpose of this note is to show that the solution of the problem just mentioned is a negative one.

We consider algebras $\mathfrak{A}_{p,t}$ defined in the following way: p and t are positive integers; $\mathfrak{A}_{p,t}$ is constituted by the set A_t of all non-negative integers less than t, and by the operation $\oplus_p$ defined by the following formula:

$$(hp + m) \oplus_p (kp + n) = mp + n$$

where h, k, m, and n are non-negative integers, $m < p$ and $n < p$. In particular, $u \oplus_1 v = 0$ for all non-negative integers u and v. In what follows we shall be interested in only those algebras $\mathfrak{A}_{p,t}$ in which $p \mid t$ and $p^2 \leqq t$; we denote by K the class of all such algebras. As is easily seen, in every algebra $\mathfrak{A}_{p,t}$ of K the set of all elements A_t is closed under the operation $\oplus_p$ and contains an element, in fact 0, which is idempotent under this operation. The following two theorems express the fundamental properties of the algebras of the class K with respect to cardinal multiplication.

Presented to the Society, November 27, 1948; received by the editors September 27, 1948.

[1] B. Jónsson and A. Tarski, *Direct decompositions of finite algebraic systems*, Notre Dame Mathematical Lectures, No. 5, Notre Dame, Indiana, 1947. The notations of this monograph are applied in the present note, except that the relation of isomorphism between two algebras is denoted by the symbol $\cong$.