MEASURABLE TRANSFORMATIONS

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1. **Introduction.** The purpose of this paper is to review the progress made in the study of measurable and measure preserving transformations during the last 17 years. The interest of mathematicians in this subject was aroused at the end of 1931 by von Neumann's and Birkhoff's proofs of their respective versions of the ergodic theorem [**8**, **9**, **101**].[1] It was very quickly recognized that the proper general framework for von Neumann's mean ergodic theorem lay in the direction of Hilbert spaces and Banach spaces, whereas the extent of generality suitable to Birkhoff's theorem was to be found in the concept of a measure space. A measure space is a set possessing no intrinsic algebraic, analytic, or topological structure—all that is necessary is that a concept of measurability and a numerical measure be defined in it. Perhaps the best known nontrivial example of a measure space is one which, to be sure, has many essential non measure theoretic properties, but which may, nevertheless, be considered typical of measure spaces in general—namely the closed unit interval $X=[0, 1]$. For the sake of definiteness I shall begin the discussion by considering a one-to-one transformation T of this space X onto itself, such that, for every measurable subset E of X, both TE and $T^{-1}E$ are measurable and $\mu(E)=\mu(TE)=\mu(T^{-1}E)$ (where μ denotes Lebesgue measure in X). In much of what follows the space X and the transformation T can be replaced by more general spaces and transformations respectively. I shall indicate some of these generalizations in what might be called the geometric direction (that is, generalizations that retain something like an underlying measure space and a transformation acting on it), but I shall not enter at all into the analytic generalizations which constitute the current theory of the mean ergodic theorem.

2. **Asymptotic properties.** The problems that were first treated, and that are still of interest and importance, are connected with the behavior of the sequence $\{T^n\}$ of powers of T. One of the first results in this direction is the Poincaré recurrence theorem, which asserts that, for every measurable set E and for almost every point x in E,

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[1] Bold face numerals refer to the bibliography at the end.