AN INEQUALITY RELATED TO THE ISOPERIMETRIC INEQUALITY

L. H. LOOMIS AND H. WHITNEY

In this note we shall prove the following theorem.

THEOREM 1. *Let m be the measure of an open subset O of Euclidean n-space, and let $m_1, \cdots, m_n$ be the $(n-1)$-dimensional measures of the projections of O on the coordinate hyperplanes. Then*

$$m^{n-1} \leqq m_1 m_2 \cdots m_n. \tag{1}$$

Note that for n-dimensional intervals with faces parallel to the coordinate hyperplanes, (1) holds with the equality sign.

With any reasonable definition of the $(n-1)$-dimensional measure s of the boundary of O, $s \geqq 2m_i$ for each i, so that (1) gives

$$m^{n-1} \leqq s^n/2^n; \tag{2}$$

this is the isoperimetric inequality, without the best constant. Since the proof of the isoperimetric inequality with the best constant is difficult,[1] and since its applications do not necessarily require the best constant, our elementary proof of the theorem may be of some interest.

We first reduce the problem to a combinatorial one, in the following theorem.

THEOREM 2. *Let S be a set of cubes from a cubical subdivision of n-space; let S_i be the set of $(n-1)$-cubes obtained by projecting the cubes of S onto the ith coordinate hyperplane. Let N and N_i be the numbers of cubes in S and S_i respectively. Then*

$$N^{n-1} \leqq N_1 N_2 \cdots N_n. \tag{3}$$

Assuming Theorem 2, we prove Theorem 1 as follows. Given $\epsilon > 0$, choose a cubical subdivision of n-space into cubes of side δ, with δ so small that if S is the set of cubes interior to O forming the set $\bar{S}$, $\mu(O - \bar{S}) < \epsilon$ (μ = measure). Then

$$[\mu(\bar{S})]^{n-1} = N^{n-1}\delta^{n(n-1)} \leqq (N_1\delta^{n-1}) \cdots (N_n\delta^{n-1}) \leqq m_1 \cdots m_n,$$

and since ϵ is arbitrary, (1) follows.

Received by the editors July 31, 1948.

[1] See E. Schmidt, *Über das isoperimetrische Problem in Raum von n Dimensionen*, Math. Zeit. vol. 44 (1939) pp. 689–788.