CONVEXITY THEOREMS

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In spite of the generality of our title, we do not intend to give here a survey of all convexity theorems. Most of them, like the three circles theorem of Hadamard, are too classical to be commented on here. We shall confine ourselves to Marcel Riesz's convexity theorem, which is one of the very powerful tools of modern analysis, and to certain of its recent extensions.

1. **Marcel Riesz's theorem.** Let

$$f = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

be a bilinear form, where the constants a_{ij} are real or complex, and the variables x_i, y_j are essentially supposed to be complex. Let $\alpha \geqq 0$, $\beta \geqq 0$ be real numbers, and denote by $M(\alpha, \beta)$ the maximum of $|f|$ under the conditions

$$\sum_1^m | x_i |^{1/\alpha} \leqq 1, \qquad \sum_1^n | y_j |^{1/\beta} \leqq 1.$$

(If $\alpha = 0$ the first condition means that $|x_i| \leqq 1$ for $i = 1, 2, \cdots, m$, and the same remark applies to the second condition.) Then $\log M(\alpha, \beta)$ is a convex function of α, β in the quadrant $\alpha \geqq 0$, $\beta \geqq 0$; in other words if $0 < t < 1$ and if

$$\alpha = \alpha_1 t + \alpha_2(1 - t), \qquad \beta = \beta_1 t + \beta_2(1 - t),$$

then

$$M(\alpha, \beta) \leqq M^t(\alpha_1, \beta_1) M^{1-t}(\alpha_2, \beta_2).$$

This is M. Riesz's fundamental theorem. (See M. Riesz [5][1] and a different proof in Paley [4]; see also a generalization of the theorem in L. C. Young [11].) M. Riesz's argument proved the convexity only in the triangle $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha + \beta \geqq 1$. The extension to the whole quadrant is due to Thorin [9]. We shall not give the proof of the theorem here, since we intend to sketch later on the proof of a more general result. Let us only point out that if we restrict the

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[1] Numbers in brackets refer to the references cited at the end of the paper.