APPROXIMATION BY CURVES OF A UNISOLVENT FAMILY

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The aim of this paper is to extend theorems on the best approximation of a given function by a polynomial of a given degree, or more generally by a linear combination of given functions, first to the case where the approximating function is to be taken from a more general family of functions, which satisfies the requirements of continuity, solvence and unisolvence, to be specified below, and secondly to a class of related geometrical problems, for which that of approximating a curve, or six given points, by an ellipse may serve as an example. The best approximation is said to be furnished by that curve for which the maximum distance between corresponding points of the approximating and the approximated curves is as small as possible.[1]

Let (S) be a family of curves S represented by one-valued functions $y=S(x)$, $-1\leqq x\leqq 1$. The family (S) is assumed to be n-parametric, solvent, unisolvent, and continuous; more explicitly we assume:

1. *Solvence*: for any n values $x_1, \cdots, x_n$ with $-1\leqq x_1<\cdots<x_n\leqq 1$ and arbitrary real numbers $y_1, \cdots, y_n$ there exists a function S of (S) with $S(x_i)=y_i$, $i=1, \cdots, n$;

2. *Unisolvence*: only one such function exists, in the extended sense that not only, for any two different functions S_0 and S_1 of (S), S_0-S_1 has less than n roots (zeros), but also that this is true if any root x with $|x|<1$ for which S_0-S_1 does not change sign between $x-\epsilon$ and $x+\epsilon$ is counted as two roots;

3. *Continuity*: $S(x)=S(x; x_1, \cdots, x_n; y_1, \cdots, y_n)$ is a continuous function of $x, y_1, \cdots, y_n$.

It follows that there cannot exist $n+1$ values $-1\leqq x_0<\cdots<x_n\leqq 1$ for which S_0-S_1 has "alternating signs," that is, is alternatingly non-negative and non-positive.

For any curve S, $\sigma=\sigma(S)$ shall denote the supremum of the values

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[1] For the approximation by linear systems of functions see S. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variables réelle*, Paris, 1926, Chap. 1. The approximation of systems of points in space and to some extent also of functions of several variables by linear functions and other polynomials is considered by P. Kirchberger, *Ueber Tchebychefsche Annaeherungsmethoden*, Math. Ann. vol. 57 (1903) pp. 509–540. It would be desirable to extend the method of the present note to the case of several independent variables.

According to Kirchberger loc. cit. p. 510, the approximation with least maximal deviation was first considered by Poncelet, and more systematically by Chebyshev.