# THE WEDDERBURN PRINCIPAL THEOREM FOR ALTERNATIVE ALGEBRAS

R. D. SCHAFER

Except for a generalization of the so-called Wedderburn principal theorem, the structure theory of alternative algebras over an arbitrary field is as complete as that for associative algebras. It is our purpose here to fill this one gap in the alternative theory.

1. **The principal theorem.** A non-associative algebra $\mathfrak{A}$ of order $n$ over an arbitrary field $\mathfrak{F}$ is called *alternative* in case

$$ax^2 = (ax)x, \qquad x^2a = x(xa)$$

for all $a$, $x$ in $\mathfrak{A}$. It is clear that associative algebras are alternative.

The most famous examples of alternative algebras which are not associative are the so-called Cayley-Dickson algebras of order 8 over $\mathfrak{F}$. Let $\mathfrak{Z}$ be an algebra of order 2 over $\mathfrak{F}$ which is either a separable quadratic field over $\mathfrak{F}$ or the direct sum $\mathfrak{F}\oplus\mathfrak{F}$. There is one automorphism $z\rightarrow\bar{z}$ of $\mathfrak{Z}$ (over $\mathfrak{F}$) which is not the identity automorphism. The associative algebra $\mathfrak{Q}=\mathfrak{Z}+u\mathfrak{Z}$ with elements

$$q = z_1 + uz_2, \qquad z_i \text{ in } \mathfrak{Z}, \tag{1}$$

and multiplication defined by

$$(z_1 + uz_2)(z_3 + uz_4) = (z_1z_3 + \beta z_4\bar{z}_2) + u(\bar{z}_1z_4 + z_3z_2) \tag{2}$$

for $\beta\neq 0$ in $\mathfrak{F}$ is called a *quaternion algebra*. For $q$ in the form (1), the correspondence

$$q \rightarrow \bar{q} = \bar{z}_1 - uz_2 = t(q) - q \tag{3}$$

is an involution of $\mathfrak{Q}$. The Cayley-Dickson algebras $\mathfrak{C}=\mathfrak{Q}+v\mathfrak{Q}$ are obtained by repetition of this process: the elements of $\mathfrak{C}$ are

$$x = q_1 + vq_2, \qquad q_i \text{ in } \mathfrak{Q}, \tag{4}$$

and multiplication in $\mathfrak{C}$ is defined by

$$(q_1 + vq_2)(q_3 + vq_4) = (q_1q_3 + \gamma q_4\bar{q}_2) + v(\bar{q}_1q_4 + q_3q_2) \tag{5}$$

for $\gamma\neq 0$ in $\mathfrak{F}$, where $q\rightarrow\bar{q}$ is the involution (3) of $\mathfrak{Q}$.

Most of our knowledge of alternative algebras is due to M. Zorn.[1]

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[1] See references [6], [7], [9]. Numbers in brackets refer to the references cited at the end of the paper.