A THEOREM ON DIFFERENCE POLYNOMIALS

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We shall prove the following analogue of a theorem of Kronecker's: *Let $\mathcal{J}$ be a difference field*[1] *containing an element t which is distinct from its transforms of any order. Every perfect ideal in the ring $\mathcal{J}[y_1, \cdots, y_n]$ has a basis consisting of $n+1$ difference polynomials.*

The corresponding theorem for differential polynomials was proved by J. F. Ritt.[2] We shall follow in all but details a brief and elegant proof of the theorem for differential equations which he has given recently.[3]

Let Λ be a perfect ideal in $\mathcal{J}[y_1, \cdots, y_n]$. We know[4] that Λ has a finite basis $A_1, \cdots, A_r$. We adjoin to $\mathcal{J}[y_1, \cdots, y_n]$ the $(n+1)r$ additional variables u_{ij}, $i=1, \cdots, n+1; j=1, \cdots, r$, and consider the polynomials[5]

$$L_i = \sum_{j=1}^{r} u_{ij}A_j, \qquad i = 1, \cdots, n+1. \tag{1}$$

We shall show that there exists a nonzero polynomial H in the u_{ij} only, which vanishes for all solutions of the system (1) which do not annul every A_i. Our theorem will follow immediately from the existence of H. For I showed in my Dissertation[6] that the presence of t in $\mathcal{J}$ implies that we may assign to the u_{ij} values α_{ij} in $\mathcal{J}$ which do not annul H. Let L_i become M_i, $i=1, \cdots, n+1$, when the u_{ij} are

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[1] For a definition of the terms used in the statement of this theorem the reader is referred to J. F. Ritt and H. W. Raudenbush, *Ideal theory and algebraic difference equations*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 445–452. The reader's attention is also called to the paper *Complete difference ideals* by J. F. Ritt, Amer. J. Math. vol. 63 (1941) p. 681, in which the definition of the term "difference ideal" is modified. It is the latter definition which is now in use. The symbol $\mathcal{J}[y_1, \cdots, y_n]$ denotes the ring of all difference polynomials in the unknowns $y_1, \cdots, y_n$, whose coefficients lie in the difference field $\mathcal{J}$.

[2] J. F. Ritt, *Differential equations from the algebraic standpoint*, Amer. Math. Soc. Colloquium Publications, vol. 14, pp. 50–58.

[3] To appear in the forthcoming revised edition of *Differential equations from the algebraic standpoint*. This shorter proof does not furnish all details of the theorem established in the paper mentioned in footnote 2.

[4] This is one of the principal results of the paper of Ritt and Raudenbush referred to in footnote 1.

[5] We use the term "polynomial" as an abbreviation for "difference polynomial."

[6] *Manifolds of difference polynomials*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 133–172. This paper is referred to below as M.D.P.