INTEGRAL EXTENSIONS OF A RING

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Introduction. Let R be a commutative ring with a unit element and let $a, b \subset R$.

DEFINITIONS. (1) a and b are said to be coprime in R if $\tau \subset R$, τ / a, τ / b implies $\tau / 1$.

(2) A ring R' is called an integral extension of R if $R' \supset R$ and $a = br$, $a, b \subset R$, $r \subset R'$ implies there exists an element $\bar{r} \subset R$ such that $a = b\bar{r}$.

(3) a and b are said to be absolutely coprime if they are coprime in every extension R' of R. In this paper it is shown that to every set of ideals of a commutative ring there exists an extension of the ring such that every ideal of the set is the intersection of the ring and a principal ideal of its extension. This is the main result and is given in Theorem 2. In a particular case of Theorem 2 it is shown in Theorem 1 that $a, b \subset R$ are absolutely coprime if and only if there exist elements $x, y \subset R$ such that $ax + by = 1$.

Similar results for algebraic integers are well known [1].[1] In the special case where the domains considered are completely integrally closed and the ideals have finite bases, a different extension fulfilling the conditions of Theorem 2 was obtained by Kronecker [2].

The extension of R. An extension of R, in the sense of this paper, may be obtained in the following manner. We first form the ring $R(u)$ by adjoining to R the elements u^n, $n = \pm 1, \pm 2, \cdots$, transcendental over R and such that $u^n a = au^n$, $a \subset R$. Let $\mathfrak{a}$ be the ideal generated by the set A of elements $a, b, \cdots$ of R. Then the subring R' of $R(u)$ consisting of all "polynomials"

$$a_{-m}u^{-m} + a_{-m+1}u^{-m+1} + \cdots + a_{-1}u^{-1} + a_0 + a_1 u + \cdots + a_n u^n$$

with $a_i \subset R$ and $a_{-r} \subset \mathfrak{a}^r$, $r > 0$, is an integral extension of R for $R' \supset R$. Moreover if $c = dh$, $c, d \subset R$, $h \subset R'$, then

$$h = e_{-m}u^{-m} + \cdots + e_{-1}u^{-1} + e_0 + e_1 u + \cdots + e_n u^n$$

$e_i \subset R$, $e_{-r} \subset \mathfrak{a}^r$. Multiplying by d we have

$$c = de_{-m}u^{-m} + \cdots + de_{-1}u^{-1} + de_0 + de_1 u + \cdots + de_n u^n.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.