GENERALIZED CONVEX FUNCTIONS AND SECOND ORDER DIFFERENTIAL INEQUALITIES

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1. **Introduction.** A well known theorem states that a necessary and sufficient condition in order that the twice differentiable function $y(x)$, $a<x<b$, be convex is that $y''\geqq 0$. The condition $y''>0$ is sufficient for the strict convexity of y.

In the present paper we show that if convexity is taken in the generalized sense of E. F. Beckenbach [1, 2],[1] a differential characterization of the above type can be obtained. As a particular case of a general theorem concerning second order differential inequalities we obtain a recent result of S. Tchaplygin, V. N. Petrov and J. E. Wilkins [3] concerning linear differential inequalities.

2. **Generalized convexity.** Let $\{F(x)\}$ be a family of real functions of the real variable x defined for $a<x<b$ and such that:

(1) Each member of the family is a continuous function of x.

(2) Given in the xy-plane two arbitrary points (x_1, y_1), (x_2, y_2) such that $a<x_1<x_2<b$, there is a unique member of the family passing through these two points, that is, such that its graph passes through these two points.

A function $\phi(x)$, $a<x<b$, is said to be *convex relative to the family* $\{F(x)\}$—*a sub-*$\{F(x)\}$ *function* in Beckenbach's notation—if, for arbitrary x_1, x_2 such that $a<x_1<x_2<b$, the member of the family, $F_{12}(x)$, which pasess through $[x_1, \phi(x_1)]$, $[x_2, \phi(x_2)]$ is such that

$$\phi(x) \leqq F_{12}(x), \qquad x_1 \leqq x \leqq x_2. \tag{3}$$

If we have

$$\phi(x) < F_{12}(x), \qquad x_1 < x < x_2, \tag{4}$$

we say that $\phi(x)$ is *strictly convex relative to the family* $\{F(x)\}$ *or else that it is a strictly sub-*$\{F(x)\}$ *function.*

The ordinary convexity is obtained if we take as the family $\{F(x)\}$ the linear functions $mx+n$.

3. **An auxiliary theorem.** $\phi(x)$ being a sub-$\{F(x)\}$ function and $a<x_0<b$ we have proved elsewhere [4] the following theorem.

THEOREM 1. *There exist* $D(x)\in\{F(x)\}$, $E(x)\in\{F(x)\}$ *such that*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.