COMBINATORIAL HOMOTOPY. II

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1. Introduction. This paper is concerned with problems of realizability, which were discussed in (I) (i.e. [1]).[1] In studying the realizability of chain mappings we use the system of relative homotopy groups, $\rho_n = \pi_n(K^n, K^{n-1})$, of a (connected) complex, K, where $n \geqq 1$ and $\rho_1 = \pi_1 = (K^1)$. The chain groups are defined as the relative homology groups, $C_n = H_n(\tilde{K}^n, \tilde{K}^{n-1})$, where $\tilde{K}$ is a universal covering complex of K. The groups ρ_n appear to be more useful than C_n in problems concerning geometrical realizability. On the other hand the chain groups are convenient in studying concrete problems. Moreover they provide a means of applying Theorem 3 in (I). A large part of the paper deals with the relations between the two systems of groups.

The system of relative homotopy groups, $\{\rho_n\}$, is a "group system," as defined in [8]. It is a special kind of group system because each group is "free" in one of three different senses. More precisely, ρ_1 is a free group, ρ_n is a free $\pi_1(K)$-module if $n > 2$ and ρ_2 is what we call a free crossed module. These conditions of freedom are important in realizability problems. We tentatively describe $\{\rho_n\}$ as a *homotopy system*. It is essentially the same as what was called a "natural system" on p. 1216 of [3], redefined in terms of relative homotopy groups and free crossed modules.

2. Crossed modules. It will be convenient to have a name for groups with the algebraic properties of relative homotopy groups, and to have proved some lemmas concerning them. We shall call such a group a *crossed module*, or, more explicitly, a *crossed γ-module* or a *crossed (γ, d)-module.*[2] By this we mean an additive, but not necessarily commutative, group, ρ, which is related as follows to a multiplicative group γ:

(2.1) (a) *ρ admits γ as a group of operators.*[3]

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Anne Cobbe has pointed out to me that a crossed (γ, d)-module determines a Q-kernel, with $Q = \gamma/d\rho$ (see [10]), and that any Q-kernel has a representation as a crossed module. I learn from Saunders MacLane that crossed modules, under a different name, are defined in a forthcoming sequel to [10].

[3] I.e. to each $x \in \gamma$ corresponds an automorphism, $x: \rho \rightarrow \rho$, such that $x'(xa) = (x'x)a$ and $xa = a$ if $x = 1$ $(x, x' \subset \gamma, a \in \rho)$. We allow trivial operators (i.e. elements $x \in \gamma$ such that $xa = a$ for every $a \in \rho$) other than $1 \in \gamma$.