ON FIBERING SPHERES BY TORUSES

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1. Introduction. In this note we answer a question which was recently asked by H. Hopf, namely whether there exist fiber decompositions of the n-dimensional sphere S^n, or of the n-dimensional Euclidean space E^n, with an s-dimensional torus T^s (the product of s factors S^1) as fiber. The answers are given in Theorems 1 and 2 below. The fiber decompositions are understood in the sense of fiber bundle [1],[1] with the group of all homeomorphisms as structure group, that is, we are given three spaces X, F and M (the bundle, the fiber and the base space), and a map $\phi: X \rightarrow M$, the projection; and for every point p of M there is given a neighborhood V_p of p and a homeomorphism of $\phi^{-1}(V_p)$ with the product $V_p \times F$, such that $\phi^{-1}(q)$ is mapped onto $q \times F$ for every point $q \in V_p$. The sets $\phi^{-1}(p)$ are called the fibers.

2. Results. We now consider the case where X is either S^n or E^n $(n>0)$, and where F is T^s $(s>0)$. We shall establish the following results:

THEOREM 1. *A fiber decomposition (in the sense of* §1*), with a (locally finite) polyhedron as base space, of the n-sphere* S^n *with the s-torus* T^s *as fiber exists if and only if n is odd and* $s=1$.

THEOREM 2. *For no n and s does there exist a fiber decomposition (in the sense of* §1*), with a (locally finite) polyhedron as base space, of Euclidean n-space* E^n *with the s-torus* T^s *as fiber.*

REMARK. We show in §7 (a) that Theorem 1 holds also if arbitrary separable metric spaces (instead of polyhedra only) are admitted as base spaces. The corresponding statement for Theorem 2 can be made only modulo a theorem concerning singular homology groups, which is probably true, but for which, as far as the authors are aware, there is no proof in the literature.

We note for the if part of Theorem 1 that there exists a well known fiber decomposition of S^{2l+1} with circles T^1 as fiber; the base space is the complex projective space K^l (of dimension $2l$) (see [2, p. 55]).

The proofs of Theorems 1 and 2 are based on the consideration of

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[1] Numbers in brackets refer to the list of references at the end of the paper.