ON THE GROUPS OF REPEATED GRAPHS

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In a recently published paper[1] Kagno showed that Pappus' graph, consisting of the 9 vertices A, B, C, D, E, F, G, H, I, and the 27 arcs AD, AE, AF, AG, AH, AI, BD, BE, BF, BG, BH, BI, CD, CE, CF, CG, CH, CI, DG, DH, DI, EG, EH, EI, FG, FH, FI, has a group of order 1296 which may be generated by the following set of eight substitutions: (ABC), (AB), (DEF), (DE), (GHI), (GH), $(ADG)(BEH)(CFI)$, $(AD)(BE)(CF)$.[2] Kagno's proof of this fact (Theorem 5)[1] is straightforward, but somewhat lengthy, and it seems of interest to note that this theorem follows at once from a more general and almost self-evident theorem on the groups of repeated graphs, if we apply to Pappus' graph the following lemma, also due to Kagno:[1] "If G' is the complement of G, then G and G' have the same group."[3] Indeed the complement Π' of Pappus' graph contains the 9 arcs AB, AC, BC, DE, DF, EF, GH, GI, HI; hence Π' is not connected, but consists of three triangles (or complete 3-points) ABC, DEF, GHI; that is, Π' is a threefold repeated triangle. To such a repeated graph we can apply the following theorem, which is of interest in itself apart from the use made of it here.

THEOREM. *If G is a connected graph of n vertices, having no simple loops, with a group $\mathfrak{H}$ of order h, and if Γ is the graph consisting of m copies G_1, G_2, $\cdots$, G_m of the same graph G, then the group of Γ is Pólya's "Gruppenkranz" $\mathfrak{S}_m[\mathfrak{H}]$, that is, the group of order $m!h^m$ and degree mn, whose substitutions may be described briefly as follows:*[4] *Let*

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[1] I. N. Kagno, *Desargues' and Pappus' graphs and their groups*, Amer. J. Math. vol. 69 (1947) pp. 859–862.

[2] It may be remarked that the same group might be generated by other sets of fewer elements, for example, by the following one containing only 3 substitutions: $(ABC)(DE)$, $(ADG)(BEH)(CFI)$, $(AD)(BE)(CF)$.

[3] Here the complement G' of a graph G (without loops) is to be defined as follows: Let $I^p_q = I^q_p = 1$, if and only if the vertices p and q are joined by an arc, otherwise let $I^p_q = I^q_p = 0$. Now, if for any pair of vertices p, q in G, $I^p_q = 1$, then in G' let $I^p_q = 0$, and if $I^p_q = 0$ in G, then in G' let $I^p_q = 1$. In other applications of this lemma some difficulty may arise from the fact that the complement of a graph may contain isolated points; for example, the complement of a triangle (or complete 3-point) consists only of three isolated points with no arcs connecting them.

[4] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. vol. 68 (1937) p. 178. The same groups have also been studied by other authors. We mention only the following papers:

Wilhelm Specht, *Eine Verallgemeinerung der symmetrischen Gruppe*, Schriften des