# SPACES OF CONTINUOUS FUNCTIONS

S. B. MYERS

Let $X$ be a completely regular topological space, $B(X)$ the Banach space of real-valued bounded continuous functions on $X$, with the usual norm $\|b\| = \sup_{x\in X}|b(x)|$. A subset $G\subset B(X)$ is called completely regular (c.r.) over $X$ if given any closed subset $K\subset X$ and point $x_0\in X-K$, there exists a $b\in G$ such that $b(x_0)=\|b\|$ and $\sup_{x\in K}|b(x)|<\|b\|$. A topological space $X$ is completely regular in the usual sense if and only if $B(X)$ is c.r. over $X$.

A Banach space $B$ is said to act completely regularly (c.r.) on $X$ if $B$ is equivalent to a closed linear subspace of $B(X)$ which is c.r. over $X$. It is known [6][1] that if $X$ is compact,[2] a closed linear subspace of $B(X)$ c.r. over $X$ determines the topology of $X$. By this is meant that if $X_1$ and $X_2$ are compact, and a Banach space $B$ acts c.r. on both $X_1$ and $X_2$, then $X_1$ is homeomorphic to $X_2$. If $B$ acts c.r. on $X$ (compact or not), $X$ is homeomorphically imbeddable in the surface of the unit sphere in $B^*_w$, the conjugate space to $B$ under the weak-* topology, and for each $b\in B$ and $x\in X$ we have the formula $b(x)=\inf_{t\in T}[\|b+t\|-\|t\|]$, where $T=\{t\in B\mid t(x)=\|t\|\}$.

If we weaken the definition of complete regularity so that $G$ is c.r. over $X$ means that for every closed set $K\subset X$ and point $x_0\in X-K$ there is a $b\in G$ such that $b(x_0)=\|b\|$, $\sup_{x\in K} b(x)<\|b\|$, then a closed linear subspace of $B(X)$ c.r. over $X$ does not necessarily determine the topology of $X$. For example, if $X$ consists of just two points, $x_1$ and $x_2$, then the subspace $G$ of $B(X)$ consisting of all $b\in B(X)$ such that $b(x_1)=-b(x_2)$ is c.r. over $X$ according to the weakened definition, yet it is equivalent to the space $B(Y)$, where $Y$ consists of a single point.

Proper closed linear subspaces of $B(X)$ which are c.r. over $X$ exist in general for both compact and non-compact $X$, and may contain the constant functions. This is in contrast to the situation when $B(X)$ is made into a normed ring (Banach algebra) $R(X)$ or into a Banach lattice $L(X)$; if $X$ is compact, no proper closed subring of $R(X)$ containing the constant functions can be c.r. over $X$ [**8**], and no proper closed sublattice of $L(X)$ containing the constant functions can be c.r. over $X$ [**4**].

Since topological properties of $X$ must be reflected in metric and

Presented to the Society, February 28, 1948; received by the editors April 5, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] "Compact" means bicompact and Hausdorff.