ON SONINE'S INTEGRAL FORMULA AND ITS GENERALIZATION

HSIEN-YÜ HSÜ

1. **Introduction.** Let $J_\nu(x)$ denote Bessel function of order ν and let $\nu > -1/2$. The value of the integral

$$(1.1) \qquad S(\nu;\, a,\, b,\, c) = \int_0^\infty J_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1-\nu} dx,$$

where a, b, c are positive parameters, was evaluated by Sonine [8, p. 411][1] and given by his formula

$$(1.2) \qquad S(\nu;\, a,\, b,\, c) = \frac{2^{\nu-1}\Delta^{2\nu-1}}{\Gamma(1/2)\Gamma(\nu + 1/2)(abc)^\nu} \quad \text{or} \quad 0,$$

according as a triangle can, or cannot, be constructed with sides a, b, c. The area of the triangle is denoted by Δ. Derivation of (1.2) may be established by means of a limiting process from either Dougall's integral formula or Dougall's expansion formula, both of which involve the product of three ultra-spherical polynomials [5, pp. 379–382]. It has further been remarked [5, p. 375] that a proof of Sonine's theorem (1.2) can also be given by a procedure similar to that employed for the proof of the latter of Dougall's results by making use of Hankel's inversion formula.

In this paper, (a) we shall show the content of the previous remark with detailed explanations. (b) By use of the result (2.7) and Sonine's formula (1.2) a generalization of Sonine's integral (3.6) can be obtained. (c) Similarly, a generalization of Dougall's expansion formula may also be derived, which will contain (3.6) as a limiting case. And (d) we shall further give an explicit expression of the generalization of Dougall's integral formula, whence by means of a limiting process we also get the same generalized Sonine's integral. For $\nu = 0$, the well known Nicholson result [8, p. 414] can be displayed as a special case of (3.6).

2. **Derivation of Sonine's integral (1.2) by means of Hankel's inversion formula.** We shall assume that the area of the triangle as constructed by the given lengths a, b, c is greater than zero. For the special case when the area Δ is zero, then we may replace the expres-

Received by the editors March 6, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.