A SUBSTITUTE FOR THE PICONE FORMULA

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It has been known for a number of years that the calculus of variations affords a powerful tool for the study of the oscillation of solutions of self-adjoint differential systems.[1] It is the purpose of the present paper to demonstrate how it can be used to replace and extend the Picone theorem.[2]

Consider the pair of self-adjoint differential equations

$$\frac{d}{dx}(ru') + pu = 0, \tag{1.1}$$

$$\frac{d}{dx}(r_1u') + p_1u = 0, \tag{1.2}$$

where, for definiteness, it is assumed that $r(x)$, $r_1(x)$, $r'(x)$, $r_1'(x)$, $p(x)$, $p_1(x)$ are continuous with r and r_1 positive on the interval $a \leqq x \leqq b$. With these equations we associate the functional identity

$$\begin{aligned}\int_a^b [(r - r_1)u'^2 + (p_1 - p)u^2]dx \\ = ruu'\Big|_a^b - \int_a^b u[(ru')' + pu]dx - \int_a^b [r_1u'^2 - p_1u^2]dx.\end{aligned} \tag{2}$$

The proof of (2) requires simply the integration by parts of the term ru'^2 in its left-hand member. In what follows it is convenient to *admit* to our discussion functions[3] $u(x)$ of class C' on the interval $a \leqq x \leqq b$ which vanish at a and b.

Repeated use is made of the following well known result.[4]

LEMMA. *If there exists an admissible curve $y = y(x)$ along which*

$$J = \int_a^b (Ry'^2 - Py^2)dx < 0, \tag{3}$$

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[1] See, for example, Morse [2, Chap. IV]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Cf. Bôcher [1, p. 54], Ince [1, p. 225].

[3] A function is said to be of class C' on an interval $a \leqq x \leqq b$ if it is continuous and has a continuous derivative on $a \leqq x \leqq b$.

[4] Cf. Morse [2, chap. 2].