ON THE REALITY OF ZEROS OF BESSEL FUNCTIONS

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We shall present some observations on the reality of zeros of Bessel functions of real order, that is, functions satisfying the differential equation

$$z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2)y = 0 \tag{1}$$

with ν real. The two linearly independent solutions $J_\nu(z)$ and $Y_\nu(z)$ may be defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r}}{r!\Gamma(\nu + r + 1)} \tag{2}$$

and

$$\begin{aligned} Y_\nu(z) &= \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi} \quad \text{for } \nu \text{ not an integer,} \\ Y_n(z) &= \lim_{\nu \to n} Y_\nu(z) \quad \text{for integers } n. \end{aligned} \tag{3}$$

J_ν and Y_ν are in general many-valued functions of z. If in (2) we replace z by the positive real variable x and use the principal value of $(x/2)^\nu$, a real valued function, $J_\nu(x)$, is obtained. Substituting $J_\nu(x)$ for $J_\nu(z)$ in (3) gives a real function $Y_\nu(x)$.

All branches of any Bessel function, $B(z)$, can be obtained by analytic continuation of a function

$$B(x) = (a + ib)J_\nu(x) + (h + ik)Y_\nu(x),$$

where a, b, h, and k are real numbers. In particular let $B(x, m)$ stand for the result of continuing $B(x)$ through an angle of $m\pi$ along a circle with center at the origin. Restricting m to be an integer, it can be shown[1] that

$$\begin{aligned} B(x, m) = {} & [(aC - bS - 2kS \cot \nu\pi)J_\nu(x) + (hC + kS)Y_\nu(x)] \\ & + i[(bC + aS + 2hS \cot \nu\pi)J_\nu(x) + (kC - hS)Y_\nu(x)] \end{aligned}$$

where $C = \cos m\nu\pi$ and $S = \sin m\nu\pi$. Each real (positive or negative) zero on any branch of the analytic function $B(z)$ is a zero of $B(x, m)$

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[1] See G. N. Watson, *A treatise on the theory of Bessel functions*. p. 75.