AN INTEGRATION SCHEME OF MARÉCHAL

J. ERNEST WILKINS, JR.

The French physicist Maréchal [1][1] has invented a mechanical integrator for studying the distribution of light in an optical image. This integrator approximates a double integral $\int_0^{2\pi}\int_0^R f(r, \phi) r dr d\phi$ by a line integral $2\pi a\int_C f(r, \phi)ds$ extended over that portion of an archimedean spiral

$$C: \qquad r = a\phi$$

which lies inside the circle $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$. The validity of this procedure when $f(r, \phi)$ is continuous (as it always is in the case of the integrals determining distribution of light in an optical image) was taken for granted by Maréchal when a is small. It is the purpose of this note to justify Maréchal's approximation by proving the following theorem.

THEOREM. *If $f(r, \phi)$ is continuous on $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$ and is periodic with period 2π in ϕ, then*

$$\lim_{a=0} 2\pi \int_0^R f(r, r/a)(a^2 + r^2)^{1/2} dr = \int_0^{2\pi} \int_0^R f(r, \phi) r dr d\phi. \tag{1}$$

Let us define

$$\begin{aligned} P_N(r, \phi) &= \frac{1}{2N\pi} \int_0^\pi \{f(r, \phi + u) \\ &\qquad + f(r, \phi - u)\} \sin^2 (Nu/2) \csc^2 (u/2) du, \\ a_{nN}(r) &= \frac{1}{2\pi} (1 - |n|/N) \int_0^{2\pi} f(r, \phi) e^{-in\phi} d\phi. \end{aligned} \tag{2}$$

Then it is known from the theory of $(C, 1)$ summability of Fourier series that

$$\begin{aligned} P_N(r, \phi) &= \sum_{n=-(N-1)}^{n=N-1} a_{nN}(r) e^{in\phi}, \\ \lim_{N=\infty} P_N(r, \phi) &= f(r, \phi) \end{aligned} \tag{3}$$

uniformly on $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$. For each positive ϵ we can therefore

Received by the editors February 19, 1948.

[1] Numbers in brackets refer to the reference cited at the end of the paper.