ON THE MEAN MODULUS OF AN ANALYTIC FUNCTION

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Throughout this paper $f=f(z)$ will denote an analytic function of the complex variable z in the open unit circle $|z|<1$. The circle $C(r)$, on which $|z|=r$, of radius $r\geqq 0$ about the origin $z=0$ lies in the region of analyticity of f provided $r<1$. For every positive real parameter t $(0<t<\infty)$ the mean of order t of the modulus of f on the circle $C(r)$ is defined as

$$(1) \qquad M_t(r;f)=\left[\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^t d\theta\right]^{1/t}.$$

For fixed f and r this mean modulus $M_t(r;f)$ as a function of t is continuous, nonnegative, nondecreasing, and is bounded above by the maximum modulus of f on $C(r)$ [1, 2].[1] Therefore the limit of $M_t(r;f)$ exists as $t\rightarrow 0$ and $t\rightarrow\infty$. This limit is defined to be the mean modulus of f on $C(r)$ of order 0 and of order ∞ respectively. It may be shown that the mean modulus of order 0 is the geometric mean of the modulus of f on $C(r)$, which is simply evaluated by Jensen's formula, and that the mean modulus of order ∞ is the maximum modulus of f on $C(r)$ [1, 2]. Thus $M_t(r;f)$ is defined for all parameters t in the compact infinite interval $0\leqq t\leqq\infty$.

For fixed f and t $(0\leqq t\leqq\infty)$ the mean modulus $M_t(r;f)$ as a function of r in the interval $0\leqq r<1$ is continuous, nonnegative, nondecreasing, and, except for the limiting parameters 0 and ∞, possesses a continuous derivative with respect to r [1, 3]. Moreover, its logarithm is a nondecreasing convex function of $\log r$ (for $t=\infty$ this is the Hadamard three-circle theorem) [1, 3].

We shall be concerned here with the convexity of the mean modulus $M_t(r;f)$ as a function of r. Let $T(f)$ be the set of all parameters t in the compact infinite interval $0\leqq t\leqq\infty$ such that $M_t(r;f)$ is a convex function of r in the interval $0\leqq r<1$. Since $M_t(r;f)$ is continuous with respect to the parameter t and since any function which is the limit of convex functions is also convex, the set $T(f)$ is a closed and hence compact subset of the parameter interval $0\leqq t\leqq\infty$. The set $T(f)$ need not, however, coincide with the entire parameter interval and indeed

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[1] Numbers in brackets refer to the references cited at the end of the paper.