SEMILATTICES AND A TERNARY OERATION IN MODULAR LATTICES

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Before discussing the subject matter proper it is necessary to introduce the following:[1]

LEMMA 1. *The inequality*

$$(1)\quad \{(x \cap (y \cap (v \cup z))\} \cup (v \cap z) \subseteq v \cup (y \cap (x \cup z)) \cup (x \cap z)$$

is identically satisfied in any lattice.

PROOF. $x \cap y \cap (v \cup z) \subseteq x \cap y \subseteq (x \cup z) \cap y \subseteq v \cup (y \cap (x \cup z))$,

$$v \cap z \subseteq v \subseteq v \cup (y \cap (x \cup z))$$

and from these two inequalities follows

$$(x \cap y \cap (v \cup z)) \cup (v \cap z) \subseteq v \cup (y \cap (x \cup z))$$
$$\subseteq v \cup (y \cap (x \cup z)) \cup (x \cap z).$$

For purposes of facility of expression the concept of *semilattice* is here introduced following Klein-Barmen [1]:[2]

DEFINITION 1. A semilattice L_s is a partially ordered system in which a relation $x\sigma y$ is defined which satisfies

S1: For all x, $x\sigma x$,

S2: If $x\sigma y$ and $y\sigma x$, then $x=y$,

S3: If $x\sigma y$ and $y\sigma z$, then $x\sigma z$,

and in which any two elements x and y have a greatest lower bound or meet xmy.

It then follows that xmy or any binary operation xoy which is closed, idempotent, commutative and associative defines, by means of the convention that $x\sigma y$ means $xmy=x$ or $xoy=x$, a semilattice L_s in which xmy or xoy is the greatest lower bound of x and y.

LEMMA 2. *The ternary operation*

$$(2)\quad [x, t, y] = (x \cap (t \cup y)) \cup (t \cap y) = (x \cup (t \cap y)) \cap (t \cup y)$$

on the elements of a modular lattice L is closed and is an idempotent and

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[1] The author is indebted to Garrett Birkhoff for the proof of Lemma 1, and for helpful criticism.

[2] Numbers in brackets refer to the bibliography at the end of the paper.