ON THE HOMOTOPY TYPE OF ANR'S

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1. Introduction. If X and Y are any spaces and if $f\colon X\to Y$ and $g\colon Y\to X$ are maps such that $gf\simeq 1$, then g is called a *left homotopy inverse* of f and f a *right homotopy inverse*[1] of g. In this case we shall say that Y *dominates*[2] X. If Y dominates X and Z dominates Y then it is easily verified that Z dominates X. If g is both a right and left homotopy inverse of f it is called a *homotopy inverse* of f and f will be called a *homotopy equivalence*. Thus the assertion that $f\colon X\to Y$ is a homotopy equivalence claims that X and Y are of the same homotopy type and, moreover, that f has a homotopy inverse.

Two maps, $f_0, f_1\colon X\to Y$ are said (cf. [**1**, pp. 49, 50] and [**2**, p. 344]) to be n-homotopic if, and only if, $f_0\phi\simeq f_1\phi$ for every map, $\phi\colon P\to X$, of every (finite) polyhedron, P, of at most n dimensions. By an *n-homotopy inverse* of a map, $f\colon X\to Y$, or an *n-homotopy equivalence* we mean the same as a homotopy inverse or a homotopy equivalence with homotopy replaced by n-homotopy throughout the definition.

By a CR-space we shall mean a connected compactum, which is an ANR (absolute neighborhood retract). Any CR-space, X, is dominated by a finite simplicial complex [**5**, Theorems 12.2, 16.2, pp. 93, 99], even if its dimensionality is infinite. We shall use ΔX to denote the minimum dimensionality of all (finite, simplicial) complexes which dominate X. Then $\Delta X\leqq \dim X$ and we may think of ΔX as a kind of "quasi-dimensionality," noticing, however, that ΔX may be less than $\dim X$, even if X is itself a finite polyhedron.

Let X, Y be CR-spaces, and let $N=\max(\Delta X, \Delta Y)$. Let $f\colon X\to Y$ be a given map and let $f_n\colon \pi_n(X)\to\pi_n(Y)$ be the homomorphism induced by f. If f is a homotopy equivalence then f_n is an isomorphism onto for each $n\geqq 1$. In §3 below we prove a sharper theorem than the converse, namely:

THEOREM 1. *If $f_n\colon\pi_n(X)\to\pi_n(Y)$ is an isomorphism onto for each $n=1,\cdots,N$, then $f\colon X\to Y$ is a homotopy equivalence.*[3]

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[1] Cf. [**1**]. Numbers in brackets refer to the references cited at the end of the paper.

[2] In this case the homomorphisms $H_n(Y)\to H_n(X)$ induced by $g\colon Y\to X$ are all onto, likewise the induced homomorphisms $\pi_n(Y)\to\pi_n(X)$, assuming X, Y to be arcwise connected. In fact $H_n(Y)$, or $\pi_n(Y)$ $(n\geqq 2)$, may be represented as the direct sum of $H_n(X)$, or $\pi_n(X)$, and the kernel of this homomorphism.

[3] If X and Y are of the same homotopy type, then each dominates the other and $\Delta X=\Delta Y$. Theorem 1 is formulated with a view to applications in which it is possible to calculate separate upper bounds for ΔX, ΔY (for example, $\dim X$, $\dim Y$).