RECURSION AND DOUBLE RECURSION

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1. Introduction. We shall apply the results of PRF[1] to construct by double recursion two functions which are not themselves primitive recursive, but which are related in interesting ways to the class of primitive recursive functions. In a sense, this note is a revised version of a paper by Rózsa Péter,[2] much simplified by the use of PRF.

Let Sx denote the successor of x. We shall say that a function G_nx of two variables n and x is defined by a double recursion from certain given functions, if

(1) G_0x is a given function of x.

(2) $G_{Sn}0$ is obtained by substitution from G_nz (considered as a function of z) and from given functions.

(3) $G_{Sn}Sx$ is obtained by substitution from the number $G_{Sn}x$, from G_nz (considered as a function of z), and from given functions.

It is clear that if the given functions are primitive recursive, then G_nx is a primitive recursive function of x for each fixed n. However, as we shall see, G_nx need not be a primitive recursive function of n and x.

In §2, we shall show that the double recursion

$$G_0x = Sx, \qquad G_{Sn}0 = G_n1, \qquad G_{Sn}Sx = G_nG_{Sn}x$$

defines a function G_nx which *majorizes* all primitive recursive functions of one variable in the following sense: *If Fx is a primitive recursive function of x, then there exists a number n such that*

$$Fx < G_nx$$

for all x. It is also shown that G_nx is an increasing function of n, so that

$$Fx < G_xx$$

for all sufficiently large x. It follows that G_xx is not a primitive recursive function of x, and hence that G_nx is not a primitive recursive

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[1] R. M. Robinson, *Primitive recursive functions*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 925–942.

[2] R. Péter, *Konstruktion nichtrekursiver Funktionen*, Math. Ann. vol. 111 (1935) pp. 42–60.