$\mathcal{J}\langle\eta_1, \cdots, \eta_n\rangle$ is contained in a liouvillian extension of $\mathcal{J}$ of the required type. It follows that $\eta_1, \cdots, \eta_n$ is a fundamental system of solutions of $L(y)$, and (see PV, §25) $\mathcal{J}\langle\eta_1, \cdots, \eta_n\rangle$ is itself a liouvillian extension of $\mathcal{J}$ of the required type.

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ON SOME EXAMPLES IN THE THEORY OF POWER SERIES

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Fabry,[1] Hardy,[2] S. Bernstein[3] and Carleman[4] discovered that for any $\delta>0$ there exist power series $\sum_0^\infty a_\nu z^\nu$ which are continuous for $|z|\leqq 1$ and for which the series $\sum|a_\nu|^{2-\delta}$ diverges. An elegant example is provided by the series

$$f(z) = \sum_{n=2}^{\infty} \frac{e^{icn\log n}}{n^{1/2}\log^\beta n} z^n, \qquad c \neq 0, \beta > 1, \tag{1}$$

which is continuous for $|z|\leqq 1$ (and even uniformly convergent there);[5] another example for the Carleman singularity, explicitly

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[1] E. Fabry, *Ordre des points singuliers de la série de Taylor*, Acta Math. vol. 36 (1913) pp. 69–194, esp. p. 103.

[2] G. H. Hardy, *A theorem concerning Tayor's series*, Quarterly Journal of Pure and Applied Mathematics vol. 44 (1913) pp. 147–160. In these two papers it was shown that if $\delta>0$ the series $f_2(z)=\sum_1^\infty e^{i\nu^{1-\delta}}z^\nu/\nu^{1-\delta}$ is uniformly convergent for $|z|<1$ and in Hardy's paper remark was made (p. 157) upon H. Bohr's problem on constructing a power series which is uniformly but not absolutely convergent for $|z|\leqq 1$.

[3] S. Bernstein, C. R. Acad. Sci. Paris (1914). He gave interesting cosine-polynomials $H(x)=\sum_{\nu=1}^{p-1} b_\nu \cos \nu x$ (p prime $\equiv 1$ mod 4) with the properties $|H(x)|\leqq 1$ and $\sum_{\nu=1}^{p-1}|b_\nu|=(p-1)/p^{1/2}$ which contains the seeds of the Carleman-singularity and were indeed the basis of Carleman's own construction.

[4] T. Carleman, *Über Fourier Koefficienten einer steigen Funktionen*, Acta. Math. vol. 41 (1918) pp. 337–384. Here is asserted explicitly and proved for the first time the existence of a continuous $f_3(x)\sim\sum(a_\nu \cos \nu x+b_\nu \sin \nu x)$ with $\sum_1^\infty(|a_\nu|^{2-\delta}+|b_\nu|^{2-\delta})=\infty$, δ arbitrarily small: the existence of a continuous power series with the same property seems to be explicitly mentioned at the first time by Sidon; see footnote 9.

[5] See G. H. Hardy and J. E. Littlewood (*Some problems of Diophantine approximation: A remarkable trigonometrical series*, Proc. Nat. Acad. Sci. U.S.A. vol. 2 (1916) pp. 583–586), who considered only the functions $f_1(z)=\sum_{n=2}^\infty (e^{icn\log n}/n^{1/2+\alpha})z^n$, $c\neq 0$, $0<\alpha<1$ (the divergence of the $(2-\delta)$th power of the moduli of the coefficients is not explicitly mentioned there). Series (1) seems to have been discussed for the first time by Zygmund in his book on trigonometric series. He uses there (in a simplified form) an argument due to Hille (*Note on a power series considered by Hardy and Littlewood*, J. London Math. Soc. vol. 4 (1929) pp. 176–182) and based on the application of Van der Corput's estimates to sums of the form $\sum e^{2\pi i g(n)}$.