A NOTE ON HILBERT'S NULLSTELLENSATZ

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In a recent paper, O. Zariski[1] has given a very simple proof of Hilbert's "Nullstellensatz." We give here another proof which while slightly longer is still more elementary.

Let K be an algebraically closed field. We consider a system of conditions

$$(1)\qquad \begin{gathered} f_1(x_1, x_2, \cdots, x_n) = 0, \qquad f_2(x_1, x_2, \cdots, x_n) = 0, \\ \cdots, f_r(x_1, x_2, \cdots, x_n) = 0; \\ g(x_1, x_2, \cdots, x_n) \neq 0 \end{gathered}$$

where $f_1, f_2, \cdots, f_r$, and g are polynomials in n indeterminates $x_1, x_2, \cdots, x_n$ with coefficients in K. The theorem states that *if the conditions* (1) *cannot be satisfied by any values x_i of K,*[2] *a suitable power of g belongs to the ideal* $(f_1, f_2, \cdots, f_r)$.[3]

PROOF. Let k be the number of x_j which actually appear in $f_1, f_2, \cdots, f_r$ and let x_i be the x_j of this kind with the smallest subscript. Denote by l the number of f_ρ in which x_i actually appears. Let m be the smallest positive value which occurs as degree in x_i of one of the f_ρ.[4] Now define a partial order for the different systems (1) using a lexicographical arrangement. If (1*) is a second system of the same type as (1) and if k^*, l^*, and m^* have the corresponding significance, we shall say that (1*) is *lower* than (1) if either $k^* < k$, or $k^* = k$ and $l^* < l$, or $k^* = k$, $l^* = l$, and $m^* < m$.

Suppose now that Hilbert's theorem is false. Then there exist systems (1) which are not satisfied by any values x_j in K, and for which no power of g lies in $(f_1, f_2, \cdots, f_r)$. Choose such a system (1) taking it as low as possible. Then for all systems (1*) lower than (1) the theorem will hold.

If k, l, m have the same significance as above, one of the f_ρ, say

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[1] Bull. Amer. Math. Soc. vol. 53 (1947) pp. 362–368.

[2] If we wish to formulate the theorem for arbitrary fields K as it is done in Zariski's paper, we have to consider a system of values $x_1, x_2, \cdots, x_n$ belonging to extension fields of finite degree over K. If no such system satisfies the conditions (1), the same conclusion can be drawn. The same proof can be used.

[3] We do not use anything from the theory of ideals except the notation $(f_1, f_2, \cdots, f_r)$ for the set of all polynomials of the form $P_1f_1+P_2f_2+ \cdots +P_rf_r$, $P_j \in K[x_1, x_2, \cdots, x_n]$, and facts which are immediate consequences.

[4] The numbers k, l, m do not depend on g.