TOPOLOGICAL RINGS

IRVING KAPLANSKY

I. INTRODUCTION

1. The literature. Topological rings were apparently first defined in van Dantzig's thesis [13], [14].[1] Of course innumerable instances of topological rings had been studied earlier: notably Hensel's p-adic numbers and their abstraction by Kürschak [54] to fields with valuations. A very considerable literature has since grown up around valuation theory and its applications to algebraic geometry, algebraic functions, and algebraic number theory. Van Dantzig's thesis inaugurated a second stream of investigation: locally compact rings. During the present decade much study has been devoted to Banach algebras (normed rings) in work begun by Gelfand and his colleagues, though foreshadowed by Stone [75], Nagumo [60], and Hebroni [32]. This address is divided into three parts more or less reflecting these three lines of investigation: topological division rings, locally compact rings, and normed algebras.

Besides papers actually referred to in the address, the bibliography includes most of the relevant contributions of the last decade. In certain respects there is no pretense of completeness: for example, there are only a few of the papers on valuations. Particular attention should perhaps be called to the reports of Köthe [52] and Lorch [55], and to the somewhat less closely related reports of Dunford [19], Taylor [78], and Hyers [34].

2. Definitions. By a topological ring we mean a ring and a Hausdorff space in which $a-b$ and ab are jointly continuous in a and b. This is substantially the definition of van Dantzig, except that he imposed a countability restriction. A brief account of the fundamentals is presented in Bourbaki [10].

By a normed algebra we mean a normed linear space which is at the same time an algebra satisfying

$$(1) \qquad \|ab\| \leqq \|a\| \, \|b\|.$$

If one merely assumes joint continuity of ab, one can pass to an equivalent norm satisfying (1). A Banach algebra is a complete

[1] Numbers in brackets refer to the bibliography at the end of the paper.