NOTE ON THE SPECTRAL REPRESENTATION OF A BOUNDED NORMAL MATRIX

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Every bounded normal (infinite) matrix N which possesses a bounded reciprocal may be written as a product of its polar factors as $N=PU=UP$, where P is positive definite and U is unitary. Using the corresponding spectral representations for P and U, a spectral representation of N may be obtained by means of Stieltjes integrals. Also, the cartesian factorization $N=H_1+iH_2$, where H_1 and H_2 are Hermitian and commutable, together with the spectral representations of H_1 and H_2 may be employed to obtain a spectral representation of N (cf. [1, 6][1] for references). It seems more natural to proceed directly by using the result on the moment problem for distribution functions in more than one dimension. The spectral representation of N is a consequence of the theorem below, which is the two-dimensional analogue of one employed by M. H. Martin [2] in obtaining the spectral representation of a bounded Hermitian matrix. The theorem holds in general for any finite number of bounded Hermitian matrices which commute with each other, but we consider the case of two such matrices for simplicity.

Notation. Let x, y denote one-column vectors with an infinity of (complex) components x_i, y_i, $i=1, 2, \cdots$. An infinite matrix A is said to be bounded if the least upper bound (l.u.b.) of the set of numbers $|y^*Ax|$ is finite ($y^*=\bar{y}'$, the conjugate transpose of y), where x and y range independently over the unit Hilbert sphere, that is, $|x|=(\sum_{i=1}^{\infty}|x_i|^2)^{1/2}=1$, $|y|=(\sum_{i=1}^{\infty}|y_i|^2)^{1/2}=1$. The matrix H is said to be Hermitian if $H=H^*$. For Hermitian matrices x^*Hx is real and

$$\underset{|x|=|y|=1}{\text{l.u.b.}} \; |y^*Hx| = \underset{|x|=1}{\text{l.u.b.}} \; |x^*Hx|.$$

If H is Hermitian and the greatest lower bound (g.l.b.) of the set of numbers x^*Hx is non-negative, that is,

$$\underset{|x|=1}{\text{g.l.b.}} \; x^*Hx \geqq 0,$$

then H is said to be non-negative definite. A positive definite matrix, P, is a Hermitian matrix satisfying

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[1] Numbers in brackets refer to references cited at the end of the paper.