ON THE DENSITY OF SOME SEQUENCES OF INTEGERS

P. ERDÖS

Let $a_1 < a_2 < \cdots$ be any sequence of integers such that no one divides any other, and let $b_1 < b_2 < \cdots$ be the sequence composed of those integers which are divisible by at least one a. It was once conjectured that the sequence of b's necessarily possesses a density. Besicovitch[1] showed that this is not the case. Later Davenport and I[2] showed that the sequence of b's always has a logarithmic density, in other words that $\lim_{n\to\infty} (1/\log n) \sum_{b_i \leqq n} 1/b_i$ exists, and that this logarithmic density is also the lower density of the b's.

It is very easy to see that if $\sum 1/a_i$ converges, then the sequence of b's possesses a density. Also it is easy to see that if every pair of a's is relatively prime, the density of the b's equals $\prod(1-1/a_i)$, that is, is 0 if and only if $\sum 1/a_i$ diverges. In the present paper I investigate what weaker conditions will insure that the b's have a density. Let $f(n)$ denote the number of a's not exceeding n. I prove that if $f(n) < cn/\log n$, where c is a constant, then the b's have a density. This result is best possible, since we show that if $\psi(n)$ is any function which tends to infinity with n, then there exists a sequence a_n with $f(n) < n \cdot \psi(n)/\log n$, for which the density of the b's does not exist. The former result will be obtained as a consequence of a slightly more precise theorem. Let $\phi(n; x; y_1, y_2, \cdots, y_n)$ denote generally the number of integers not exceeding n which are divisible by x but not divisible by $y_1, \cdots, y_n$. Then a necessary and sufficient condition for the b's to have a density is that

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leqq n} \phi(n; a_i; a_1, a_2, \cdots, a_{i-1}) = 0. \tag{1}$$

The condition (1) is certainly satisfied if $f(n) < cn/\log n$, since

$$\frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leqq n} \phi(n; a_i; a_1 \cdots a_{i-1}) < \frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leqq n} \left[\frac{n}{a_i}\right]$$

$$< \sum_{n^{1-\epsilon} < m \log m < n} \frac{c'}{m \log m} = O(\epsilon) + O\left(\frac{1}{n}\right).$$

Received by the editors April 28, 1947, and, in revised form, September 5, 1947.

[1] Math. Ann. vol. 110 (1934–1935) pp. 336–341.

[2] Acta Arithmetica vol. 2.