SET FUNCTIONS AND SOUSLIN'S HYPOTHESIS

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1. Introduction. It is known[1] that Souslin's hypothesis[2] is *implied* by the existence of a nontrivial outer measure on every field of sets satisfying certain conditions. We shall here prove that Souslin's hypothesis is *equivalent* to the existence, on a wide class of fields of sets, of set-functions of a certain type. The axiom of choice is assumed, but not the continuum hypothesis.

Instead of working with fields of sets, it is more convenient to use the equivalent notion of a (finitely additive) Boolean algebra, E.[3] We say that $x, y \in E$ are *disjoint* if $x \wedge y = o$, and that they *intersect* otherwise. A set S of elements of E will be called a *Souslin system* if it satisfies the following three postulates:

(1) $S \not\ni o$, and whenever $s, s' \in S$, then either $s \wedge s' = o$, or $s \geqq s'$, or $s' \geqq s$.

(2) If $A \subset S$ consists of pairwise disjoint elements only, then A is(at most) countable.

(3) If $A \subset S$ is such that every two of its elements intersect, then A is countable.

Souslin's hypothesis is known to be equivalent to the assertion that every Souslin system is countable.[4]

THEOREM. *Souslin's hypothesis is true if and only if there exists, on each non-atomic Boolean algebra E satisfying the countable chain condition, a real-valued function f such that* (i) $x \geqq y \rightarrow f(x) \geqq f(y)$, (ii) $f(x) = 0 \leftrightarrow x = o$, *and* (iii) *given* $x \in E - (o)$ *and* $\epsilon > 0$, *there exists* $y \in E - (o)$ such that $y < x$ and $f(y) < \epsilon$.

2. "If." Suppose an uncountable Souslin system exists. Then, as easily follows from [2, §7], there exists a complete Boolean algebra E, satisfying the countable chain condition, and an uncountable Souslin system $S \subset E$ having the following additional properties:

(4) $S = \bigcup S_\alpha$, where α ranges over all countable ordinals, and the elements of each S_α are pairwise disjoint.

(5) If $\alpha < \beta$, then for each $s_\beta \in S_\beta$ there exists an s_α ($\in S_\alpha$) such that $s_\alpha > s_\beta$.

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[1] See [2]; in the case of a measure, the result is due to K. Gödel. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Souslin, Fund. Math. vol. 1 (1920) p. 223.

[3] See [2] for notations, and so on.

[4] This follows from [3], together with some results in [1].