ON UNIQUE INVARIANT MEASURES

HENRY M. SCHAERF

1. Statement of the problems. Let S be a σ-field[1] of subsets (measurable sets) of an abstract group G. What can be said about the structure of S if there is a unique measure defined on S and invariant under the group operation? What are the conditions for the uniqueness of an invariant measure? These are the problems studied in this note by means of a simple lemma.

2. Definitions and results.

DEFINITION 1. "*Measure*" means in this paper a non-negative, countably additive function of the set $X \in S$ such that G is not of measure 0 and is the union of a sequence of measurable sets of finite measure. For any two measures m and n we denote by $S_{m,n}$ the σ-field of subsets of $G \times G$, defined so as to allow the application of the generalized theorem of Fubini [1, p. 87].[2]

DEFINITION 2. A measure m is called *invariant* if $A \in S$, $g \in G$ implies $gA \in S$ and $m(gA) = m(A)$. An invariant measure is called *unique* if it differs from any other invariant measure only by a multiplicative constant.

FUNDAMENTAL ASSUMPTION. It is assumed that $g \in G$, $A \in S$ implies $Ag \in S$ and that any two invariant measures m, n satisfy the following *condition* M_1: The transformation $[(x, y) \rightarrow (y^{-1}x, y)]$ sends every set $A \times G$ with $A \in S$ into a set of $S_{m,n}$.

DEFINITION 3. A measurable set A is called *almost congruent by finite* (*resp. denumerable*) *partition* with the measurable set A' if there is a finite (resp. infinite) sequence of disjoint measurable subsets A_k of A with $m(A - \bigcup_k A_k) = 0$ and a corresponding sequence of elements g_k of G such that the sets $g_k^{-1}A_k$ are disjoint subsets of A' and $m(A' - \bigcup_k g_k^{-1}A_k) = 0$.

The answer to our first problem is given by the following theorem.

THEOREM 1. *If the measure is unique invariant then any measurable set A, whose measure is not greater than that of a measurable set B or equal to it, is almost congruent by finite or denumerable partition with some measurable subset of B or with B, respectively.*

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[1] That is, a class of sets containing G and closed under complementation and the formation of countable unions.

[2] Numbers in brackets refer to the bibliography at the end.