A NOTE ON CONVERGENCE IN LENGTH

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1. Introduction. Let I be a closed linear interval $a_0 \leqq t \leqq b_0$. Let $\mathfrak{x}(t) = (x(t), y(t), z(t))$, $t \in I$, represent a vector function whose three components $x(t)$, $y(t)$, $z(t)$ are of bounded variation and continuous on I. This vector function determines in Euclidean 3-space a curve $x = x(t)$, $y = y(t)$, $z = z(t)$ whose length we denote by $L(\mathfrak{x})$. By convergence in length of a sequence of such vector functions $\mathfrak{x}_n(t) = (x_n(t), y_n(t), z_n(t))$, $n = 0, 1, 2, \cdots$, is meant that $x_n(t)$, $y_n(t)$, $z_n(t)$ converge uniformly on I to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively and that $L(\mathfrak{x}_n)$ converges to $L(\mathfrak{x}_0)$. We denote by $V(f)$ the total variation on I of a scalar function $f(t)$ which is continuous and of bounded variation on I. By convergence in variation of a sequence $f_n(t)$, $n = 0, 1, \cdots$, is meant that $f_n(t)$ is continuous and of bounded variation on I for $n = 0, 1, \cdots$, that $f_n(t)$ converges uniformly on I to $f_0(t)$, and that $V(f_n) \to V(f_0)$. These concepts are due to Adams, Clarkson, and Lewy [1, 2].[1]

We are concerned here with the problem of determining conditions under which convergence in length holds. Uniform convergence on I of the components $x_n(t)$, $y_n(t)$, $z_n(t)$ to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively implies only that $\liminf L(\mathfrak{x}_n) \geqq L(\mathfrak{x}_0)$. It is also well known (see [2, 4, 5]) that convergence in length of such a sequence $\mathfrak{x}_n$ implies convergence in variation of each of the three sequences of components—and, indeed, convergence in variation of any sequence of scalar functions obtained by projecting the curves $\mathfrak{x} = \mathfrak{x}_n(t)$, $t \in I$, $n = 0, 1, \cdots$, on any line whatever. As a consequence of this we see that convergence in length of the sequence $\mathfrak{x}_n(t)$ implies convergence in variation of the sequence $c_1x_n(t) + c_2y_n(t) + c_3z_n(t)$ for arbitrary choice of the constants c_1, c_2, c_3. Convergence in variation of each of the three sequences of components is not sufficient to ensure convergence in length of the sequence of vectors (see [2]). In connection with the work of A. P. Morse [4] there arose the question as to whether convergence in length is implied by convergence in variation of every linear combination of the components. This has already been proved by Morse [4] for the case where $\mathfrak{x}_n(t)$ is of the special form $(t, y_n(t), 0)$, $n = 0, 1, \cdots$. In this note we generalize Morse's result to the parametric case. The proof is based on a generalization,

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[1] Numbers in brackets refer to the bibliography at the end of the paper.