RECURSIVE PROPERTIES OF TRANSFORMATION GROUPS. II

W. H. GOTTSCHALK

The purpose of this note is to sharpen a previous result on the transmission of recursive properties of a transformation group to certain of its subgroups. [See *Recursive properties of transformation groups*, by W. H. Gottschalk and G. A. Hedlund, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 637–641.]

Let T be a multiplicative topological group with identity e. A subset R of T is said to be *relatively dense* provided that $T=RK$ for some compact set K in T.

LEMMA 1. *If R is a relatively dense closed semi-group $(RR\subset R)$ in T, then R is a subgroup of T.*

PROOF. Suppose $r\in R$ and U is a neighborhood of e. It is sufficient to show that $r^{-1}U\cap R\neq\emptyset$. Let V be a neighborhood of e for which $VV^{-1}\subset U$ and let K be a compact set in T for which $T=RK$. There exists a finite collection F of right translates of V which covers K. Choose $k_0\in K$. Now $r^{-1}k_0=r_1k_1$ for some $r_1\in R$ and some $k_1\in K$. Again $r^{-1}k_1=r_2k_2$ for some $r_2\in R$ and some $k_2\in K$. This may be continued. Thus there exist sequences $k_0, k_1, \cdots$ in K and $r_1, r_2, \cdots$ in R such that $r^{-1}k_i=r_{i+1}k_{i+1}$ $(i=0, 1, \cdots)$. Select integers m and n $(0\leqq m<n)$ and an element V_0 of F such that $k_m, k_n\in V_0$. Now $r^{-1}k_mk_n{}^{-1} = (r^{-1}k_mk_{m+1}^{-1})(k_{m+1}k_{m+2}^{-1})\cdots(k_{n-1}k_n^{-1}) = r_{m+1}rr_{m+2}\cdots rr_n\in R$. Also $r^{-1}k_mk_n^{-1}\in r^{-1}V_0V_0^{-1}\subset r^{-1}VV^{-1}\subset r^{-1}U$. Hence $r^{-1}U\cap R\neq\emptyset$ and the proof is completed.

Now let T act as a transformation group on a topological space X. That is to say, suppose that to $x\in X$ and $t\in T$ is assigned a point, denoted xt, of X such that: (1) $xe=x$ $(x\in X)$; (2) $(xt)s=x(ts)$ $(x\in X; t, s\in T)$; (3) The function xt defines a continuous transformation of $X\times T$ into X. We assume for the remainder of the paper that x is a fixed point of X, T is locally compact and S is a relatively dense invariant subgroup of T. Let Σ denote the maximal subset of T for which $x\Sigma\subset(xS)^*$ where the star denotes the closure operator.

LEMMA 2. *The set Σ is a closed subgroup of T which contains S.*

PROOF. Obviously $\Sigma\supset S$. From $x\Sigma^*\subset(x\Sigma)^*\subset(xS)^*$ we conclude that Σ is closed. By Lemma 1 it is now enough to show that Σ is a

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